# AN ELEMENTARY TREATISE

ON THE

GEOMETRICAL AND ALGEBRAICAL INVESTIGATION

OF

# MAXIMA AND MINIMA,

BEING THE SUBSTANCE

OF

**A Course of Lectures**

*Delivered conformably to the Will of* LADY SADLER:

TO WHICH IS ADDED,

**A Selection of Propositions**

DEDUCIBLE FROM EUCLID'S ELEMENTS.

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*Second Edition, Corrected and Enlarged.*

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have been recorded, although, from my want of judgment or ability, they have not been fulfilled. I can, indeed, with great truth affirm, that I was most desirous of being able to meet your Lordship's wishes, when I applied myself to this attempt. But, besides my own deficiencies, I had to encounter an obstacle of another kind. It was difficult to discover a subject important in itself, and claiming the early attention of the mathematical student, which is not sufficiently illustrated in the Lectures constantly delivered by the Tutors of the College. This circumstance will, I trust, be accepted by your Lordship as my excuse, and it must, at the same time, be offered to the public as my apology, for not having accomplished more, nor attempted higher things.

I gladly avail myself of this occasion publicly to acknowledge the several favours which your Lordship, with the utmost kindness, has been pleased to confer upon me.

I am, my Lord,
with the sincerest gratitude and respect,
your Lordship's
most obliged
and most obedient servant,

D. CRESSWELL.

# APPENDIX I.

CONTAINING

## A SELECTION OF PROPOSITIONS

DEDUCIBLE FROM

## *THE FIRST SIX BOOKS*

OF

## Euclid's Elements.

Censemus, autem, nihil utilius ad Geometriam penitius cognoscendam haberi posse, quam hujusmodi contentio tyronis, in deducendis theorematis, vel solvendis problematis: qua fit, ut Geometria ipsa ejus animo multo altius insideat, et investigationis fontes aperiantur.

BOSCOVICH.

Univ. of California

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

# Euclid's Elements.

Geometrical Exercises are peculiarly adapted to the improvement of the chief powers of the mind; and the sole motive which has prompted the publication of the following selection of Questions, is a desire to engage the Academical Student in that employment, from which he is likely to be most benefited in the beginning of his course. It is by no means intended to recommend to him the cultivation of this department of Science, to the neglect of all others; but he is advised to make it the ground-work of his future acquirements in the Mathematics, and never to advance until he has laid that foundation well; because it is the firmest upon which the superstructure of solid mathematical knowledge can be built.

No credit whatever is expected by the author to accrue to him from the execution of this design,

unless it be of that very humble, but surely not dishonourable, kind, which belongs to an useful performance. Although some of the questions, which he has here published, are original, he is far from thinking it probable that even these have not occurred to others, at least as early as they did to himself.

His materials are, for the most part, taken from works of established reputation, both ancient and modern. Well known, as they must be to the learned, they may, however, be useful, as a collection, to the student in Geometry. They have been chosen, either as exhibiting some remarkable property of lines or figures, omitted by Euclid, or as furnishing a mere exercise of ingenuity. Some Propositions, that are very obvious, and very easy of demonstration, are purposely inserted, as best suited to the ability of beginners; and, perhaps, it may not be improper to add, that many others are the Geometrical Solutions of Problems belonging to the several branches of Natural Philosophy.

Such an arrangement has been given to the deductions, as will, in many cases, lead to at least *one* method, it would be presumption to say the *best* method, of solution. They are distributed according to the same order as the several Books of Euclid, which are most studied; and further, they are placed, each under the *last* of the propositions upon which it may be made to depend, or which need be quoted in its proof. They are not always, therefore, although they are often,

strictly speaking, corollaries of the proposition to which they are immediately subjoined. Sometimes, indeed, it is best, to make use of one of these deductions as a step in the demonstration of another, which follows it; but this is not very often the case; and when it happens to be so, the reader is most commonly apprized of it. It is only, therefore, in those particular cases, in which some of the following theorems, or problems, are, properly speaking, corollaries of the proposition from Euclid, which is placed at the head of them, that the mode of solution, intended to be pursued, is clearly intimated by the arrangement which has been here adopted. In other cases, the difficulty of the deduction, such as it is, may mainly depend upon some other antecedent proposition in Euclid's book, although that, which is referred to, be also required in the course of the demonstration. To have distinctly pointed out all the elements upon which each demonstration is made to depend, would have been to leave too little for the ingenuity of the learner to perform. If, however, this book should happen to be used, under the direction of a tutor, it would be easy for him to supply as many of these purposely suppressed references, as he may judge to be necessary.

To this second edition is annexed a specimen of propositions belonging to Natural Philosophy, of which the solutions may very well be derived from the application of Geometry to that extensive and interesting subject. These examples, it is hoped,

may serve still further to illustrate the elegancy of geometrical constructions, and also to stimulate the curiosity of the student, by the diversified and perhaps more engaging forms, in which the questions thus involved are presented to his mind. As, in solving algebraical problems, his first step is to translate the conditions of the problem into the peculiar language of analytical calculation; so here, it will be for him, in the first place, making use of the principles of Natural Philosophy, to reduce the question, under his consideration, to the substance of some geometrical proposition; which being solved, the question itself may be regarded as solved also. If he has any taste for Plane Geometry, this will be far from being a disagreeable exercise; and if he has acquired any skill in that, the most lucid of all the branches of mathematical learning, it will also be an easy task.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

# Euclid's Elements.

## BOOK I.

### Prop. IX.

(I.)

A given plane rectilineal angle being divided into any number of equal angles, to divide the half of it into the same number of angles, all equal to one another.

### Prop. X.

(II.)

From the vertex of a given scalene triangle, to draw, to the base, a straight line which shall exceed the less of the two sides, as much as it is itself exceeded by the greater.

## Prop. XI.

### (iii.)

In a straight line given in position, but indefinite in length, to find a point, which shall be equidistant from each of two given points, either on contrary sides, or both on the same side of the given line, and in the same plane with it; but not situated in a perpendicular to it.

### (iv.)

If the three sides of a given triangle be bisected, the perpendiculars drawn to the sides, from the three several bisections, shall all meet in the same point: And that point is equidistant from the three angular points of the given triangle.

### (v.)

Hence, to find a point, in a given plane, which shall be equidistant from three given points in the plane, that are not all in the same straight line.

## Prop. XVI.

### (vi.)

There cannot be drawn more than two equal straight lines, to another straight line, from a given point without it.

Cor. A circle cannot cut a straight line in more points than two.

## Prop. XVII.

### (VII.)

The perpendicular let fall from the obtuse angle of an obtuse-angled triangle, or from any angle of an acute-angled triangle, upon the opposite side, falls within that side: But the perpendicular drawn to either of the sides containing the obtuse angle of an obtuse-angled triangle, from the angle opposite, falls without that side.

### (VIII.)

If a straight line, meeting two other straight lines, makes the two interior angles on the same side of it not less than two right angles, these lines shall never meet on that side, if produced ever so far.

Cor. Two straight lines, which are both perpendicular to the same straight line, are parallel to each other.

## SCHOLIUM.

Parallel straight lines being thus defined, "Two straight lines are parallel if they be in the same plane, and a straight line drawn from any point in the one, perpendicular to either of them, be also perpendicular to the other," the 35th Definition

of the First Book of Euclid becomes a corollary to the last article; and may be cited in the demonstration of Prop. 27. Book I.; also the 29th Proposition of this Book may be proved by a *reductio ad absurdum,* without the help of the 12th axiom, which itself becomes a corollary to that proposition. Nothing seems to be wanting, to render the definition here substituted unobjectionable, but a proof, that if it be a property of two straight lines at any one point, it will also obtain at every other point of them. This proof has been given by Robert Simson in his Note upon E. 29. 1.; and it is made to depend only upon an axiom and upon the 4th and 8th Proposition of the First Book of Euclid. It has also been given, under a somewhat different form, by Borelli, in a single proposition, clearly and elegantly demonstrated; but the axiom borrowed from the Arabian mathematicians, which he premises, is, perhaps, less judiciously chosen, than that which is the foundation of Simson's proof. The great object of this substitution is to avoid the necessity of having recourse to the proposition, which Euclid has made his 12th axiom; but neither is his 35th definition itself the best that might be given; for, with the exception of the single supposition, that the two straight lines are in the same plane, it is wholly negative, and affords no practical test of the parallelism of two straight lines.

A modern editor of the Elements, of great learning and ability, has proposed the following axiom,

instead of the 12th:—"Two straight lines cannot be drawn through the same point parallel to the same straight line, without coinciding with one another." But this pre-supposes the 35th Definition of Euclid, and is, therefore, objectionable, inasmuch as that definition is itself objectionable. Besides, if it be thus stated, according to its true meaning,—"Two straight lines cannot be drawn through the same point, neither of which, when they are produced ever so far both ways, meets another straight line, given in position, but indefinite in length," it does not appear to have a much better claim to the title of axiom, than the assumption which it is intended to replace. Garnier has accordingly, in his Geometry, formally demonstrated this very proposition: It is, indeed, most certain, that the mind cannot conceive two straight lines in a state of infinite extension; which, however, it is led to attempt, by the 35th definition, and the axiom last considered.

## Prop. XX.

### (IX.)

The three sides of a triangle taken together, exceed the double of any one side, and are less than the double of any two sides.

(x.)

Any side of a triangle is greater than the difference between the other two sides.

(xi.)

Any one side of a rectilineal figure is less than the aggregate of the remaining sides.

(xii.)

The two sides of a triangle are together, greater than the double of the straight line which joins the vertex and the bisection of the base.

## Prop. XXI.

(xiii.)

If a trapezium and a triangle stand upon the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.

## Prop. XXVI.

(xiv.)

If two right-angled triangles have the three angles of the one equal to the three angles of the

other, each to each, and if a side of the one be equal to the perpendicular let fall from the right angle upon the hypotenuse of the other, then shall a side of this latter triangle be equal to the hypotenuse of the former.

(XV.)

If the sides of any given equilateral and equiangular figure of more than four sides, be produced so as to meet, the straight lines, joining their several intersections, shall contain an equilateral and equiangular figure, of the same number of sides as the given figure.

## Prop. XXVII.

(XVI.)

If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.

Cor. 1. Hence may be deduced a practical method of drawing a straight line, through a given point, parallel to a given straight line.

Cor. 2. A rhombus is a parallelogram.

## Prop. XXIX.

(XVII.)

Every parallelogram which has one angle a right angle, has all its angles right angles.

(XVIII.)

To trisect a right angle; i. e. to divide it into three equal parts.

(XIX.)

Hence, to trisect a given rectilineal angle, which is the half, or the quarter, or the eighth part, and so on, of a right angle.

## Prop. XXXI.

(XX.)

To find a point, in either of the equal sides of a given isosceles triangle, from which, if a straight line be drawn, perpendicular to that side, so as to meet the other side produced, it shall be equal to the base of the triangle.

(XXI.)

In the hypotenuse of a right-angled triangle, to find a point, the perpendicular distance of which

from one of the sides, shall be equal to the segment of the hypotenuse between the point and the other side.

(XXII.)

In the base of a given acute-angled triangle, to find a point, through which if a straight line be drawn perpendicular to one of the sides, the segment of the base, between that side and the point, shall be equal to the segment of the perpendicular, between the point and the other side produced.

(XXIII.)

From a given isosceles triangle to cut off a trapezium, which shall have the same base as the triangle, and shall have its three remaining sides equal to each other.

(XXIV.)

To draw to a given straight line, from a given point without it, another straight line which shall make with it an angle equal to a given rectilineal angle.

(XXV.)

The two sides of a triangle are, together, greater than the double of the straight line drawn from the vertex to the base, bisecting the vertical angle.

## Prop. XXXII.

### (XXVI.)

If two triangles have two angles of the one equal to two angles of the other, the third angle of the one shall also be equal to the third angle of the other.

### (XXVII.)

The angle at the base of an isosceles triangle is equal to, or is less, or greater, than the half of the vertical angle, accordingly as the triangle is a right-angled, an obtuse-angled, or an acute-angled triangle.

### (XXVIII.)

If either of the equal sides of an isosceles triangle be produced, towards the vertex, the straight line, which bisects the exterior angle, shall be parallel to the base.

### (XXIX.)

The distance of the vertex of a triangle from the bisection of it's base, is equal to, greater than, or less than the half of the base, accordingly as

the vertical angle is a right, an acute, or an obtuse angle*.

Cor. 1. If any number of triangles have a right angle for their common vertical angle, and have equal hypotenuses, the locus of the bisections of the several hypotenuses is a quadrantal arch of a circle, having the common vertex for its center, and the half of any hypotenuse for its radius.

Cor. 2. A circle described from the bisection of the hypotenuse of a right-angled triangle as a center, at the distance of half the hypotenuse, will pass through the summit of the right angle.

(XXX.)

If either of the acute angles of a given right-angled triangle be divided into any number of equal angles, then, of the segments of the base, subtending those equal angles, the nearest to the right angle is the least; and, of the rest, that which is nearer to the right angle is less than that which is more remote.

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* It is intended that this proposition should be demonstrated, *ex absurdo*, by the help of E. 32. 1. E. 5. 1. and E. 18. 1. But it is evidently deducible, with equal facility, from E. 31. 3. It will, doubtless, often happen to the reader, in other instances, as well as in this, very readily to find out another mode of proof, when he does not, at the first attempt, discover the principle of solution, intimated by the arrangement which has been adopted in this Appendix.

(XXXI.)

If either angle at the base of a triangle be a right angle, and if the base be divided into any number of equal parts, that which is adjacent to the right angle shall subtend the greatest angle at the vertex; and, of the rest, that which is nearer to the right angle shall subtend, at the vertex, a greater angle than that which is more remote.

(XXXII.)

To trisect a given finite straight line.

Cor. Hence, to inscribe a square in a given right-angled isosceles triangle.

(XXXIII.)

To describe a triangle which shall have its three sides, taken together, equal to a given finite straight line, and its three angles equal to three given angles, each to each; the three given angles being together equal to two right angles.

(XXXIV.)

If, in the sides of a given square, at equal distances from the four angular points, four other points be taken, one in each side, the figure contained by the straight lines which join them, shall also be a square.

(XXXV.)

If the opposite angles, of a quadrilateral figure be equal to each other, the figure shall be a parallelogram.

(XXXVI.)

In a given square to inscribe an equilateral triangle, having one of its angular points upon one of the angular points of the square, and its two remaining angular points one in each of two adjacent sides of the square.

(XXXVII.)

If, at the extremities of the base of a given triangle, two straight lines be drawn, both above the base, and each of them equal to the adjacent side, and making with it an angle equal to the vertical angle of the triangle; then, if two straight lines, let fall from the extremities of the two so drawn, make, with the base produced, two angles that are equal each of them to the vertical angle, they shall cut off equal segments from the base produced.

(XXXVIII.)

To inscribe a square in a given rhombus.

(XXXIX.)

If four straight lines cut each other, without including space, but so as to make three internal angles, towards the same parts, which together are less than four right angles, the two lines, which are not joined, shall meet, if produced far enough.

(XL.)

If the straight line, drawn from a point in the produced diameter of a circle to the convex circumference be equal to the half of the diameter, the angle, at the center, subtended by the *concave* circumference included between the diameter and the line so drawn, is the triple of the angle, at the center, subtended by the *convex* circumference included between the same two lines.

The converse of the proposition is also true.

Cor. Hence, if a straight line could be drawn from any point in the curve of a semi-circle to meet the diameter produced, so that the part of the line without the curve should be equal to the radius, any angle might be trisected.

(XLI.)

One of the two sides, which are about the right angle of a right-angled triangle, and the aggregate of the hypotenuse and the remaining side, being given, to construct the triangle.

## Prop. XXXIV.

### (XLII.)

The diameters of a parallelogram bisect each other.

### (XLIII.)

The diameters of an equilateral four-sided plane rectilineal figure bisect one another at right angles.

### (XLIV.)

The diameters of a rectangle are equal to one another.

### (XLV.)

If two opposite sides of a parallelogram be divided each into the same number of equal parts, the straight lines, joining the opposite points of division, shall also divide the diameter of the parallelogram into the same number of equal parts.

### (XLVI.)

To divide a given finite straight line into any given number of equal parts.

(XLVII.)

Upon a given finite straight line, as a *diameter*, to describe a square.

(XLVIII.)

Upon a given finite straight line to describe an equilateral and equiangular octagon.

(XLIX.)

If either diameter of a parallelogram be equal to a side of the figure, the other diameter shall be greater than any side of the figure.

(L.)

From a given point to draw a straight line cutting two parallel straight lines, so that the part of it, intercepted between them, shall be equal to a given finite straight line, not less than the perpendicular distance of the two parallels.

(LI.)

If, from the summit of the right angle of a scalene right-angled triangle, two straight lines be drawn, one perpendicular to the hypotenuse, and the other bisecting it, they shall contain an angle equal to the difference of the two acute angles of the triangle.

## Prop. XXXVI.

(LII.)

To bisect a parallelogram by a straight line drawn through a given point in one of its sides.

(LIII.)

A trapezium, which has two of its sides parallel, is the half of a rectangle between the same parallels, and having its base equal to the aggregate of the two parallel sides of the trapezium.

## Prop. XXXVII.

(LIV.)

A plane rectilineal figure of any number of sides being given, to find an equal rectilineal figure, which shall have the number of its sides less, or greater, by one, than that of the given figure.

(LV.)

Hence, first, to find a triangle, which shall be equal to any given plane rectilineal figure: secondly, to find a polygon of any given number of sides which shall be equal to a given triangle.

## Prop. XXXVIII.

(LVI.)

The diameters of any parallelogram divide it into four equal triangles.

(LVII.)

Of all triangles, which are between the same parallels, that which stands on the greatest base is the greatest.

(LVIII.)

The straight line, joining the vertex and the bisection of the base of any triangle, bisects every other straight line that is parallel to the base and is terminated by the two remaining sides of the triangle.

(LIX.)

Hence, if two opposite sides of a trapezium be parallel to one another, the straight line, joining their bisections, bisects the trapezium.

(LX.)

To bisect a given trapezium by a straight line drawn from any of its angles.

(LXI.)

To bisect a given triangle, by a straight line drawn through a given point in any one of its sides.

## Prop. XL.

(LXII.)

Equal triangles, which have their bases in the same straight line and which are between the same parallels, stand upon equal bases.

## Prop. XLI.

(LXIII.)

To describe a parallelogram, the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.

(LXIV.)

The two triangles formed by drawing straight lines, from any point within a parallelogram, to the extremities of either pair of opposite sides, are, together, half of the parallelogram.

(LXV.)

If two sides of a trapezium be parallel, the triangle contained by either of the other sides, and

the two straight lines drawn from its extremities to the bisection of the opposite side, is the half of the trapezium.

(LXVI.)

The triangle contained by the straight lines joining the points of the bisection of the three sides of a given triangle, is one-fourth part of the given triangle, and is equiangular with it.

(LXVII.)

Hence, if the four sides of any given quadrilateral rectilineal figure be bisected, the figure contained by the straight lines joining the several points of the bisection, shall be a parallelogram, which is the half of the given figure; also the four sides of this parallelogram shall be, together, equal to the two diagonals of the given figure.

COR. It is manifest that the straight lines, which join the opposite points of bisection of the sides of any trapezium, bisect each other.

## PROP. XLIII.

(LXVIII.)

To describe a parallelogram, which shall be of a given altitude, and equiangular with, and also equal to, a given parallelogram.

Cor. Hence, a rectangle may very readily be found, which shall be equal to a given square, and shall have one of its sides equal to a given straight line.

## Prop. XLV.

(LXIX.)

If there be any number of rectilineal figures, of which the first is greater than the second, the second than the third, and so on, the first of them shall be equal to the last together with the aggregate of all the differences of the figures.

(LXX.)

To find a rectangle, which shall have one of its sides equal to a given finite straight line, and which shall be equal to the excess of the greater of two given rectilineal figures above the less.

## SCHOLIUM.

From the twenty-second, and the forty-fifth propositions, of this first book of Euclid, may be deduced a method of surveying, planning, and measuring irregular plots of ground, which have

rectilineal boundaries: and this method, which is purely geometrical, does not require the use of any instrument constructed for the purpose of estimating the magnitudes of angles; its operations being performed, all of them, by means of a rule containing a scale of equal parts, a compass, and some standard measure of linear magnitude.

It is manifest, that any rectilineal plot of ground may be divided into triangles; and that the lengths of the sides of those several triangles may be found by the application of the standard measure, whether it be a foot, a yard, or any other standard measure of length, which has been chosen. Then, by the help of E. 22. 1, and of the scale of equal parts, an exact *plan* of the ground may be laid down on paper: and, lastly, a rectangle may (E. 45.1. Cor.) be described which shall be equal to the figure representing the plot of ground, and which shall have one of its sides equal to one, or to any given number, of the equal parts of the scale. If, therefore, one of its sides be made equal to *one* of the equal parts of the scale, it is plain that the number of such parts in the adjacent side, will shew the dimension of the plot of ground in square measure. For, it will indicate how many squares, each having one of the equal parts for its side, are contained in the rectangle that is equal to the *plan* of the ground; and so many squares, it is evident, each having the standard measure, that was used, for its side, will there be in the plot of ground itself.

If the side of the rectangle, constructed by

means of E. 45. 1, be taken any given multiple of one of the equal parts of the scale, then, if that multiple constitute any other standard measure of length, the dimension of the ground will still be found, as before, by finding how many of those multiples there are in the adjacent side; but it will be of a different denomination.

But if the multiple, assumed for one side of the equal rectangle, be not any standard measure of length, the number of equal parts contained in the adjacent side must be counted; and the product of this latter number, multiplied by the number of equal parts in the assumed side, will shew the dimension of the constructed rectangle, and of the plot of ground, also, which is required to be measured.

For, although a rectangular surface can only be measured, in a *direct* manner, by the application of some lesser standard square, to its several parts in succession, yet, since it is evident, even from inspection, that any rectangle may be divided into a number of lesser squares, equal to the product of the numbers which shew how often a side of one of those lesser squares is contained in each of the two adjacent sides of the rectangle, that direct method of measurement is never employed. Hence, no such instrument as a standard square is wanted for the measurement of surfaces, and no such instrument is in use. By means of E. 35. 1, E. 41. 1, and E. 45. 1. the mensuration of parallelograms, triangles, and other rectilineal figures, is

reduced to the mensuration of a rectangle, which is effected, in the manner already described.

Thus a parallelogram is denoted, in square measure, by the product of the number of standard equal parts in its base multiplied by the number of such parts in its altitude: and a triangle is also denoted, by the *half* of the product of the number of standard equal parts in its base multiplied by the number of such parts in its altitude.

The plan of the piece of ground, required to be measured, having been previously drawn, if the deduction from E. 37. 1, set down in this book, be had recourse to, a triangle will be obtained, that is equal to the rectilineal figure so drawn: and the operations by which this result is arrived at, will be very easily and expeditiously performed, by the help of a parallel ruler. It will then only remain, to let fall, from the vertex of the triangle, a perpendicular on its base; to measure both the perpendicular and the base; and, lastly, to take the half of the product of the resulting numbers.

The mode of planning and measuring which has here been described, is not, it is true, sufficient for all practical purposes. It contains, however, the first principles of the mensuration of plane surfaces. It has the advantage of being very simple and very easy to be understood: and it may, perhaps, afford some degree of satisfaction to the mathematical student, to consider with how small a stock of Geometry he may be enabled to solve a problem of no small utility and importance.

## PROP. XLVII.

(LXXI.)

If two triangles have two sides of the one equal to two sides of the other, each to each, and if the angles opposite to either pair of equal sides be each a right angle, the triangles shall be equal, and similar to each other.

(LXXII.)

To find a square which shall be equal to any number of given squares.

(LXXIII.)

Two unequal squares being given, to find a third square, which shall be equal to the excess of the greater of them above the less.

(LXXIV.)

If the side of a square be equal to the diameter of another square, the former square shall be the double of the latter.

(LXXV.)

In any right-angled triangle, the square which is described on the side subtending the right angle, as a *diameter*, is equal to the squares described upon the other two sides, as diameters.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

# Euclid's Elements.

## BOOK II.

### Prop. I.

(I.)

If two given straight lines be divided, each into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of the one and the several parts of the other.

### Prop. V.

(II.)

If a straight line be divided into two unequal parts, in two different points, the rectangle con-

tained by the two parts, which are the greatest and the least, is less than the rectangle contained by the other two parts; the squares of the two former parts, together, are greater than the squares of the two latter, taken together; and the difference between the squares of the former and the squares of the latter, is the double of the difference between the two rectangles.

(III.)

In any isosceles triangle, if a straight line be drawn from the vertex to any point in the base, the square upon this line, together with the rectangle contained by the segments of the base, is equal to the square upon either of the equal sides.

## Prop. VI.

(IV.)

The rectangle contained by the aggregate and the difference of two unequal straight lines is equal to the difference of their squares.

Cor. If there be three straight lines, the difference between the first and second of which is equal to the difference between the second and third, the rectangle contained by the first and third, is less than the square of the second, by the square of the common difference between the lines.

## Prop. VII.

(v.)

The square of the excess of the greater of two given straight lines above the less, is less than the squares of the two lines, by twice the rectangle contained by them.

(vi.)

The squares of any two unequal straight lines are, together, greater than twice the rectangle contained by those lines.

## Prop. VIII.

(vii.)

If a straight line be divided into five equal parts, the square of the whole line is equal to the square of the straight line, which is made up of four of those parts, together with the square of the straight line which is made up of three of those parts.

(viii.)

Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, such that the

hypotenuse, together with the less of the two remaining sides, shall be the double of the greater of those sides.

## Prop. X.

(IX.)

In any triangle, the squares of the two sides are, together, the double of the squares of half the base, and of the straight line joining its bisection and the opposite angle.

(X.)

Hence, the squares of the sides of any parallelogram are, together, equal to the squares of its diameters taken together.*

(XI.)

If either diameter of a parallelogram be equal to one of the sides about the opposite angle of the figure, its square shall be less than the square of the other diameter, by twice the square of the other side about that opposite angle.

* This proposition may also be readily deduced from E, 13. 2.

## Prop. XII.

(XII.)

If two sides of a trapezium be parallel to each other, the squares of its diagonals are, together, equal to the aggregate of the squares of its two sides, which are not parallel, and of twice the rectangle of its parallel sides.

## Prop. XIII.

(XIII.)

The square of the base of an isosceles triangle is the double of the rectangle contained by either side, and by the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the extremity of the base.

(XIV.)

If from any point, in the circumference of the greater of two given concentric circles, two straight lines be drawn to the extremities of any diameter of the less, their squares shall be, together, the double of the squares of the two semi-diameters of the two given circles.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

# Euclid's Elements.

## BOOK III.

### Prop. I.

(1.)

If two circles cut each other, the straight line joining their two points of intersection is bisected, at right angles, by the straight line joining their centers.

Cor. Hence, if a trapezium have two of its adjacent sides equal to one another, and also its two remaining sides equal to one another, its diameters bisect each other at right angles.

## SCHOLIUM.

It is evident from the above deduction, that the double of the perpendicular, let fall from the vertex of a triangle on its base, may very readily be found by the compass alone: And a reference to the Scholium in the twenty-seventh page, of this Appendix, will shew that this expedient might be of great utility in Practical Geometry, for the mensuration of the surfaces of triangles and other rectilineal figures.

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### Prop. III.

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(II.)

Through a given point within a circle, which is not the center, to draw a chord which shall be bisected in that point.

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### Prop. XIV.

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(III.)

If two isosceles triangles be of equal altitudes, and the side of the one be equal to the side of the other, their bases shall be equal.

(IV.)

Any two chords of a circle which cut a diameter in the same point and at equal angles are equal to one another.

(V.)

Through a given point, within a given circle, to draw two equal chords, making with one another an angle equal to a given rectilineal angle.

## Prop. XVI.

(VI.)

If the diameters of two circles are in the same straight line, and have a common extremity, the two circles shall touch one another.

(VII.)

To draw a tangent to a circle, which shall be parallel to a given finite straight line.

Cor. Hence, to draw a tangent to a circle, which shall make, with a given straight line, an angle equal to a given rectilineal angle.

(VIII.)

To describe a circle which shall have a given radius, and its center in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.

(IX.)

To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in another given point.

(X.)

To describe a circle, the circumference of which shall pass through a given point, and touch a given circle in another given point; the two points not lying in a tangent to the circle.

(XI.)

To describe a circle, which shall touch a given straight line in a given point, and also touch a given circle.

(XII.)

Hence, if two circles touch each other externally, to describe another circle, which shall touch the one of them in a given point, and also touch the other.

(XIII.)

To describe two circles, each having a given radius, which shall touch the same given straight line, both on the same side of it, and shall also touch each other.

(XIV.)

To describe two equal circles, each having it's diameter equal to a given straight line, each touching a given circle, and each also passing through a given point without that circle: The given straight line being greater than the shortest distance, between the given point and the circumference of the given circle.

## Prop. XVII.

(XV.)

To find a point in the diameter, produced, of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.

(XVI.)

Through a given point, either within, or without a given circle, to draw a straight line, so that

the part of it within the circle shall be equal to a given finite straight line, which is less than the diameter.

(XVII.)

To draw a tangent to a given circle, such that it's segment, contained between the point of contact, and an indefinite straight line, given in position, shall be equal to a given finite straight line.

## Prop. XVIII.

(XVIII.)

If a straight line touch the interior of two concentric circles, and be terminated both ways by the circumference of the outer circle, it shall be bisected in the point of contact.

(XIX.)

If a polygon be described about a circle, the straight lines joining the several points of contact will contain a polygon of the same number of angles as the former; and any two adjacent angles of the circumscribed figure shall be, together, the double of that angle, of the inscribed figure, which lies between them.

## PROP. XX.

(XX.)

If two chords of a given circle intersect each other, the angle of their inclination is the half of the angle at the center standing upon the aggregate, or the difference, of the arches intercepted between them, accordingly as they meet within, or without the circle.

(XXI.)

To divide a given circular arch into two parts, so that the aggregate of their chords may be equal to a given straight line, greater than the chord of the whole arch, but not greater than the double of the chord of half the arch.

(XXII.)

To divide a given circular arch into two parts, so that the excess of the chord of the one above the chord of the other, may be equal to a given straight line, less than the chord of the whole arch.

(XXIII.)

If from any given point, in the circumference of a circle, two straight lines be drawn to the ex-

tremities of a given chord, the angle which the one makes with any perpendicular to the chord, shall be equal to the angle which the other makes with the diameter of the circle that passes through the given point.

## Prop. XXI.

### (XXIV.)

The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.

Cor. This point is equidistant from the three straight lines joining the points in which the perpendiculars meet the three sides of the triangle.

### (XXV.)

If from a given point within a circle, which is not the center, straight lines be drawn to the circumference, making with each other equal angles, the two, which are nearer to the diameter passing through the given point, shall cut off a greater circumference than the two, which are more remote.

### (XXVI.)

From either of the two given points, in which two given circles intersect each other, to draw a

chord cutting the one circumference, and meeting the other, such that the part of it, contained between the two circumferences, shall be equal to a given finite straight line.

## Prop. XXII.

(XXVII.)

If two opposite angles of a trapezium be together equal to two right angles, a circle may be described about it.

(XXVIII.)

A circle cannot be described about a rhombus, nor about any other parallelogram which is not rectangular.

(XXIX.)

If from any point, in the circumference of a given circle, straight lines be drawn to the three angles of an inscribed equilateral triangle, the greatest of them shall be equal to the aggregate of the two less.

(XXX.)

The first, third, fifth, &c. angles of any polygon, of an even number of sides, which is inscribed in a given circle, are together equal to the remaining angles of the figure; any angle whatever being assumed as the first.*

(XXXI.)

To make a trapezium, about which a circle may be described, having its four sides respectively equal to four given straight lines, two of which are equal to each other, and any three together greater than the fourth; the two equal sides of the trapezium, also, being opposite to each other.

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* The Proposition may be adapted to the case of a polygon of an odd number of sides inscribed in a circle, by dividing any one of its angles into two, by a straight line drawn from the center, and reckoning the two segments of that angle, each as one of the angles of the figure; the number of angles is thus made even.

The converse of this Theorem, and the second deduction from Prop. 18, may be applied to discover the relation which the angles of a polygon must have in order that it may admit of a circle being described about it, or inscribed in it.

## PROP. XXIV.

### (XXXII.)

If upon the two greater sides of an oblong, as diameters, two semi-circles be described, lying toward the same parts, the figure contained by the two remaining sides of the oblong, and the two curve lines, shall be equal to the oblong.

## PROP. XXVI.

### (XXXIII.)

The straight lines joining the extremities of the chords of two equal arches of the same circle, toward the same parts, are parallel to each other.

### (XXXIV.)

The arches of a circle that are intercepted between two parallel chords are equal to one another.

### (XXXV.)

In equal circles the greater angle stands upon the greater circumference; whether the angles compared be at the centers or circumferences.

(XXXVI.)

If from any given point, without a circle, there be drawn two straight lines cutting the circle, then of the circumferences which they intercept, that which is the nearer to the given point is less than the other.

## Prop. XXVII.

(XXXVII.)

In equal circles, the greater of two circumferences subtends the greater angle, whether the angles compared be at the centers or circumferences.

## Prop. XXVIII.

(XXXVIII.)

In equal circles, the greater chord subtends the greater circumference.

## Prop. XXIX.

(XXXIX.)

In equal circles, the greater circumference has the greater chord.

If any equilateral and equiangular polygon be inscribed in a circle, a straight line drawn, from any of its angles, through the center of the circle, bisects the opposite side at right angles.

(XL.)

The two straight lines in a circle, which join the extremities of two parallel chords, are equal to each other.

## Prop. XXX.

(XLI.)

If from any point, in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, the other at right angles to the diameter, the squares upon these two lines are, together, the double of the square upon the semi-diameter.

## Prop. XXXI.

(XLII.)

If the chords of two arches of any the same circle cut each other at right angles, the squares of the four segments of the chords, are, together, equal to the square of the diameter.

Cor. Hence, if the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, cut

each other at right angles, the aggregate of the squares of the sides is the double of the square of the diameter of the circle.

(XLIII.)

To draw a straight line, cutting two concentric circles, so that the part of it which lies within the greater circle may be the double of the part which lies within the less.

(XLIV.)

To draw a straight line which shall touch two given circles.

(XLV.)

If the point, in which two straight lines that are perpendicular to each other meet, be applied to the circumference of a circle so that the straight lines themselves cut the circumference, the center of the circle is in the bisection of the straight line joining those two intersections.

Thus the center of a given circle may readily be found by means of the instrument called a *square*.

(XLVI.)

If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord

of the circle, that is not parallel to the diameter, the less perpendicular shall be equal to the segment of the greater contained between the circumference and the chord.

(XLVII.)

If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, they shall meet the chord, produced, in two points which are equidistant from the center.

(XLVIII.)

If upon either radius bounding a quadrantal circular arch as a diameter, a semi-circle be described, any chord of the semi-circle drawn from the center of the quadrant shall be equal to the perpendicular distance of the point, in which the chord produced meets the quadrantal arch, from the other radius.

## Prop. XXXII.

(XLIX.)

If the angle contained by two straight lines, one of which cuts a circle and the other meets it, be equal to the angle in the alternate segment of the circle, the straight line which meets, shall touch the circle.

(L.)

A straight line touching a circular arch in the bisection of that arch, is parallel to its chord.

(LI.)

If an equilateral triangle be described about a given circle, the straight lines joining the points of contact shall contain another equilateral triangle; and the side of the circumscribed triangle is the double of the side of the inscribed triangle so contained.

## Prop. XXXIII.

(LII.)

Upon a given finite straight line to describe a segment of a circle, which shall be similar to a given segment of another circle.

(LIII.)

If any equilateral rectilineal figure, of an *even* number of sides, be inscribed in a given circle, a curvilineal figure may be found that is equal to it, and that is bounded by arches of circles, each of which circles is equal to the given circle.

(LIV.)

The base, the vertical angle, and the altitude of a triangle being given, to construct the triangle.

(LV.)

To find a point in a given straight line, from which if straight lines be drawn to two given points, on the same side of the given line, they shall contain an angle equal to a given rectilineal angle.

(LVI.)

The vertical angle, the base, and the aggregate of the three sides of a triangle being given, to construct the triangle.

(LVII.)

The perimeter and the three angles of a triangle being given, to construct the triangle.

(LVIII.)

From two given points, in the circumference of a circle, to draw two equal chords of that circle, which, produced if necessary, shall make with one another an angle equal to a given rectilineal angle.

## Prop. XXXV.

(LIX.)

To produce a given straight line so that the rectangle, under the given straight line, and the part of it produced, shall be equal to a given square.

(LX.)

If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

(LXI.)

If through the given extremity of any diameter of a circle straight lines be drawn to meet an indefinite straight line without the circle, which is perpendicular to the diameter produced, the rectangles contained by the segments of these lines lying between the given point, the point in which each of them cuts the circumference again, and the indefinite line, shall be equal to each other.

(LXII.)

From the obtuse angle of an obtuse-angled triangle, to draw a straight line to the base, the square of which shall be equal to the rectangle contained by the segments, into which it divides the base.

(LXIII.)

To make a rectangle which shall be equal to a given square, and shall have its two adjacent sides, together, equal to a given straight line; the side

of the given square being less than the half of the given straight line.

## Prop. XXXVI.

### (LXIV.)

If two tangents be drawn to a given circle, from any the same point without it, they shall be equal to each other; only two tangents can be drawn to a circle from the same given point without it; and if from a given point without a circle, two equal straight lines be drawn to the convex circumference, one of which touches the circle, the other shall also touch it.

### (LXV.)

If a quadrilateral rectilineal figure be described about a circle, the angles subtended, at the center of the circle, by any two opposite sides of the figure, are, together, equal to two right angles.

### (LXVI.)

If two given straight lines touch a circle, and if any number of other tangents be drawn, all on the same side of the center, and all terminated by the two given tangents, the angles which they subtend, at the center of the circle, shall be equal to one another.

Cor. The two segments, which any two tangents, so drawn, cut off from the two given tangents, also subtend equal angles, at the center of the circle.

(LXVII.)

To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.

(LXVIII.)

If, from the bisection of any given arch of a circle, a straight line be drawn cutting the chord of that arch, or the chord produced, and the circumference also of the circle, the rectangle contained by the two parts of the straight line so drawn, the one lying between the point of bisection and the circumference, the other between the point of bisection and the chord, shall be equal to the square of the chord of half the arch.

(LXIX.)

Hence, from the bisection of a given arch of a circle, to draw a straight line, such that the part of it intercepted between the chord, or the chord produced, of the given arch and the circumference, shall be equal to a given straight line.

(LXX.)

Hence, through any given angle of a given equilateral four-sided figure, to draw a straight line terminated by the sides produced, containing the angle opposite to the given angle, which shall be equal to a given straight line.

## SCHOLIUM.

In the application of Algebra to Geometry, the equation upon which the construction of a Problem depends may, sometimes, be of more dimensions than two, and yet the solution may be obtained by means of Plane Geometry. If, therefore, the resulting equation be of an higher order than a quadratic, a trial must be made whether it be not divisible by some binomial, one of the terms of which is a divisor of the last term of the equation, and the other a similar power of the unknown quantity; it may thus, perhaps, be reduced to a quadratic equation. If it be a biquadratic, Des Cartes' method (Wood's *Alg.* Art. 329.) should be had recourse to.

Thus the last question, if the figure be a square, may be solved by a simple and elegant Geometrical construction. But, if it be treated algebraically, the resulting equation is a biquadratic.

For, let $a$ denote the side of the square, $b$ the given straight line, and $x$ and $y$ the two segments of

the sides produced, cut off by the line which is to be drawn through the angular point equal to $b$.

Then, (I.) $x^2+y^2=b^2$ (E. 47. 1.), and

(II.) $(x-a) \cdot y = ax$ (E. 4. 6)

Whence, $x^4-2ax^3+(2a^2-b^2)x^2+2ab^2x-a^2b^2=0$; which equation is reduced, by *Des Cartes*' method, to this cubic;

$y^6+(a^2-2b^2)y^4-(a^4-b^4)y^2-(a^6+2a^4b^2+a^2b^4)=0$; and $a^2+b^2$ being one of the divisors of the last term, $y^2-a^2-b^2$ is found, upon trial, to divide this equation without a remainder; the quotient is $y^4+(2a^2-b^2)y^2+(a^4+a^2b^2)=0$; which is an equation of the quadratic form, and, therefore, the question may be solved by means of Plane Geometry.

But, now, the straight line, which is to be investigated, being supposed to have been actually drawn, let $x$ be put for the segment between either of its extremities, and the angular point of the square through which it passes; from its other extremity let a perpendicular be drawn to it, so as to meet the opposite side of the square produced; and let $y$ denote the segment of the produced side of the square, lying between that angular point and the perpendicular: Then, the side of the square being denoted, as before, by $a$, and the given straight line by $b$, it follows from Ded. I. Prop. 26. B. 1, that the perpendicular is equal to $x$; and by the

help of E. 47. 1, and E. 4. 6, the following equations are easily deduced:

$$y^2 = (b-x)^2 + x^2$$
$$= b^2 - 2(b-x).x$$
$$\text{i.e. } y^2 = b^2 - 2ay;$$
$$\therefore y = \sqrt{a^2+b^2} - a.$$

And, thus, the straight line, required to be drawn, is determined by the solution of an equation, which presents itself, in the very first instance, in the common form of an adfected quadratic.

If, in the analytical investigation of this question, whether the figure be a square or a rhombus, Trigonometrical forms be introduced, the resulting equation will be a quadratic; and, as the sine of any angle is also the sine of its supplement, that equation gives four different positions of the line to be placed between the two sides of the figure produced. So that when the problem is made general, the sides must be produced *both* ways.

(LXXI.)

If two circles cut each other, and from any point, in the straight line produced, which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.

(LXXII.)

If two circles cut each other, and if two tangents drawn, one to each circle, from any point

without them, be equal, the straight line, joining the intersections of the circles, shall, if it be produced, pass through the common extremity of the equal tangents.

(LXXIII.)

The straight line which passes through the intersections of two circles, that cut one another, bisects the straight line which touches both the circles.

(LXXIV.)

Two circles being given, neither of which lies within the other, to draw a straight line, such that the tangents to the two circles, drawn from any point of the line, shall be equal to one another.

(LXXV.)

To divide a given straight line into two parts, so that the square of the one shall be equal to the rectangle contained by the other and a given straight line.

(LXXVI.)

If a given circle be cut by any number of circles, which all pass through the same two given

points without the given circle, the straight lines, joining the points of each of these intersections, are either all parallel, or all meet when produced in the same point.

(LXXVII.)

If a perpendicular be let fall from the right angle, of a right-angled triangle, on the hypotenuse, the rectangle contained by the hypotenuse and either of the segments, into which it is divided by the perpendicular, is equal to the square of the side adjacent to that segment.

(LXXVIII.)

To draw a tangent to a circle, such, that the part of it intercepted between two straight lines, given in position, but of indefinite length, shall be equal to a given finite straight line:

1st, When the indefinite straight lines both pass through the center of the circle:

2dly, When they are parallel to one another:

3dly, When they are not parallel, but are equidistant from the center.

(LXXIX.)

If two straight lines, which touch two given circles, the one touching both the circles on the one side of them, the other on the other, be cut by a third tangent, which touches the two circles on contrary sides of them, then, of the segments into which the two first tangents are thus divided, those which are alternate are equal to one another.

(LXXX.)

The perimeter, the vertical angle, and the altitude of a triangle being given, to construct the triangle.

(LXXXI.)

If from the intersection of any two tangents to a circle, any straight line be drawn cutting the chord which joins the two points of contact and again meeting the circumference, it shall be divided by the circumference and the chord into three segments, such, that the rectangle contained by the whole line and the middle part, shall be equal to the rectangle contained by the extreme parts.

(LXXXII.)

To make a rectangle which shall be equal to a given square, and have the difference between its two adjacent sides equal to a given straight line.

(LXXXIII.)

From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within, the circle, shall be equal to a given square.*

## Prop. XXXVII.

(LXXXIV.)

To describe a circle which shall touch a given straight line, and pass through two given points, both on the same side of the given line.

(LXXXV.)

To describe a circle which shall touch two given straight lines, and pass through a given point between them.

(LXXXVI.)

To describe a circle which shall touch two given straight lines, and also touch a given circle.

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* The limitation to this Problem is evident from the construction by which it is solved: The same remark may also be applied to several other problems, in this Appendix, in the enunciations of which the necessary limitation is not specified.

(LXXXVII.)

To describe a circle which shall pass through a given point, and touch both a given circle, and a given straight line.

(LXXXVIII.)

In a straight line of indefinite length, but given in position, which cuts a given circle, to find a point, from which if a straight line be drawn to touch the circle, it shall be equal to a given finite straight line.

(LXXXIX.)

To describe a circle that shall touch a given straight line, and that shall also touch two given circles, neither of which lies within the other.

(XC.)

To describe a circle which shall touch a given circle, and pass through two given points, either both without the circle, or both within it.

(XCI.)

To find a point in a straight line given in position from which if two straight lines be drawn to two given points, without the given line, they shall have, first, their difference, and, secondly, their aggregate, equal to a given finite straight line.

(XCII.)

The base and the altitude of a triangle being given, together with the aggregate or the difference, of the two remaining sides, to construct the triangle.

(XCIII.)

Three points being given, to find a fourth, from which if straight lines be drawn to the other three, two of them shall be equal, and the difference between either of these and the third shall be equal to a given straight line.

(XCIV.)

To describe a circle that shall touch three given circles, of which two are equal to one another.

(XCV.)

To find a point, in the circumference of a given circle, from which if two straight lines be drawn to two given points, without the circle, the chord joining the intersections of the lines so drawn and the circumference, shall be parallel to the straight line joining the two given points.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

# Euclid's Elements.

## BOOK IV.

### Prop. IV.

(I.)

Three straight lines being given, which, when produced, do not all three meet in the same point, and of which the middle line is not parallel to either of the others, to describe a circle which shall touch each of them.

(II.)

The three straight lines, which bisect the three angles of a triangle, meet in the same point.

(III.)

If a circle be inscribed in a right-angled triangle, the excess of the two sides, containing the right angle, above the third side, is equal to the diameter of the inscribed circle.

(IV.)

The straight line bisecting any angle of a triangle, inscribed in a given circle, cuts the circumference in a point which is equidistant from the extremities of the side opposite to the bisected angle, and from the center of a circle inscribed in the triangle.

(V.)

In a given circle, to inscribe three equal circles, touching each other and the given circle.

(VI.)

To inscribe three circles in an isosceles triangle, touching each other, and each of them touching two of the three sides of the triangle.

PROP. VI.

(VII.)

The square, inscribed in a circle, is equal to the half of the square upon its diameter.

(VIII.)

In a given circle, to inscribe a rectangle equal to a given rectilineal figure, not exceeding the half of the square upon the diameter.

(IX.)

If from any point, in the circumference of a given circle, straight lines be drawn to the four angular points of an inscribed square, the aggregate of the squares of the four lines, so drawn, shall be the double of the square of the diameter.

## Prop. VII.

(X.)

In a given circle, to inscribe four circles equal to each other, and in mutual contact with each other and the given circle.

Cor. In the same manner, four equal circles may be inscribed in a given square, touching each other and the sides of the square.

## Prop. VIII.

(XI.)

To inscribe a circle in a given rhombus.

(XII.)

To inscribe a circle in a given trapezium, of which two opposite sides are, together, equal to the other two sides taken together.

## Prop. X.

(XIII.)

Upon a given finite straight line, to describe an equilateral and equiangular decagon.

## Prop. XI.

(XIV.)

Upon a given finite straight line, to describe an equilateral and equiangular pentagon.

(XV.)

The angle of a regular pentagon exceeds a right angle by one-fifth part of a right angle; and is three times as great as the angle contained by any two sides of the figure, which are not adjacent to each other, produced so as to meet.

(XVI.)

If isosceles triangles could be described, having their angles at the base, any required multiple of the angle at the vertex, any regular rectilineal figures whatever of an *uneven* number of sides, might be inscribed in a circle; and if isosceles triangles could be described, having each of the angles

at the base to the angle at the vertex, in the ratio of any odd numbers to two, any regular figures whatever of an *even* number of sides might be inscribed in a given circle; the bases of these several triangles being the sides of the several inscribed figures.

## SCHOLIUM.

It is manifest, that, by the help of E. 9. 1, E. 2. 4, E. 6. 4, E. 11. 4, and E. 16. 4, the circumference of a circle may be divided into three, six, twelve, &c. equal parts; into four, eight, sixteen, &c. equal parts; into five, ten, twenty, &c. equal parts; and into fifteen, thirty, sixty, &c. equal parts. But the problem, announced in the last deduction, has not yet been solved geometrically: Nor has any other *general* method, depending upon plane geometry, been discovered, by which the circumference of a circle may be divided into any given number of equal parts.

## Prop. XV.

(XVII.)

The square of the side of a regular pentagon, inscribed in a given circle, is equal to the square of the side of a regular decagon, together with the

square of the side of the regular hexagon, both inscribed in that given circle.

(XVIII.)

Upon a given finite straight line, to describe an equilateral and equiangular hexagon.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

## Euclid's Elements.

## BOOK V.

### Prop. XII.

(I.)

If any number of equal ratios be each greater than a given ratio, the ratio of the sum of their antecedents, to the sum of their consequents, shall be greater than that given ratio.

### Prop. XIII.

(II.)

If the first of four magnitudes have a greater ratio to the second than the third has to the fourth,

the second shall have to the first a less ratio than the fourth has to the third.

## Prop. XVI.

(III.)

If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.

## Prop. XVII.

(IV.)

If the first, together with the second, of four magnitudes, have a greater ratio to the second, than the third, together with the fourth, has to the fourth, the first shall have a greater ratio to the second than the third has to the fourth.

## Prop. XVIII.

(V.)

If the first of four magnitudes have a greater ratio to the second than the third has to the fourth,

the first, together with the second, shall have to the second a greater ratio than the third, together with the fourth, has to the fourth.

(VI.)

If the first term of a ratio be less than the second, the ratio shall be increased by adding the same quantity to both terms; but if the first term be greater than the second, the ratio shall be diminished by adding the same quantity to both.

(VII.)

If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, and a less ratio than the first has to the second.

(VIII.)

If the first, together with the second, have to the second, a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first, a less ratio than the third, together with the fourth, has to the third.

(IX.)

If the first, together with the second, have to the third, together with the fourth, a greater ratio

than the first has to the third, then shall the second have to the fourth a greater ratio, than the first, together with the second, has to the third, together with the fourth.

## Prop. XIX.

(x.)

If any number of magnitudes be proportionals, their differences shall, also, be proportionals.

## Prop. XXII.

(xi.)

If there be three magnitudes, and other three, and if the first have a greater ratio to the second, in the former set, than the first has to the second, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the second has to the third, in the latter; then shall the first have a greater ratio to the third, in the former set, than the first has to the third, in the latter.

## Prop. XXIII.

(xii.)

If there be three magnitudes, and other three, and if the first have to the second, in the former

set, a greater ratio than the second has to the third, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the first has to the second, in the latter; then shall the first have to the third, in the former set, a greater ratio, than the first has to the third, in the latter.

## Prop. XXV.

### (XIII.)

If three magnitudes be proportionals, the two extremes are, together, greater than the double of the mean.

Cor. An arithmetic mean proportional, between two given magnitudes, is greater than a geometric mean proportional between the same two magnitudes.

### (XIV.)

If three magnitudes be proportionals, the excess of the greatest above the mean, is greater than the excess of the mean above the least of them.

### (XV.)

If there be two sets of magnitudes, the one geometric, and the other arithmetic, proportionals, and if the two first magnitudes be the same in both,

any other magnitude in the former set, shall be greater than the corresponding magnitude in the latter.

COR. The two first magnitudes, in both the sets, being the same, if the second of the geometric proportionals be greater than the second of the arithmetic proportionals, then, much more, will every other magnitude, in the former set, be greater than the corresponding magnitude in the latter.

(XVI.)

If there be two series of magnitudes, the one arithmetically proportional, the other geometrically proportional, but each having the same magnitude for its first term, and if the last term of the arithmetic series be not less than the last term of the geometric series, any other term of the former series shall be greater than the corresponding term in the latter.

# DEDUCTIONS

## FROM THE FIRST SIX BOOKS

OF

## Euclid's Elements.

## BOOK VI.

### Prop. I.

(i.)

If the bases of four rectangles be proportionals, and their altitudes be also proportionals, the rectangles themselves shall likewise be proportionals.

Cor. 1. If four straight lines be proportionals, their squares shall also be proportionals.

Cor. 2. Conversely, if four squares be proportionals, their sides shall likewise be proportionals.

### Prop. III. and A.—*Simson's Euclid.*

(ii.)

To cut a given straight line in harmonic proportion.

## Prop. IV.

### (III.)

If the base of an isosceles triangle be produced to meet a straight line drawn from the opposite angle perpendicular to either of the equal sides, that side shall be a mean proportional between the base and the half of the line which is made up of the base and of the part produced.

### (IV.)

The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon described about the circle.

### (V.)

Through a given point, within a triangle, to draw a straight line cutting the sides, either of them being produced, if necessary, so that the rectangle contained by the segments between the vertex and the cutting line, may be equal to a given square.

### (VI.)

If from a given point, without a circle, two straight lines be drawn to the concave circum-

ference, they shall be reciprocally proportional to the parts of them between the given point and the convex circumference.

(VII.)

The straight lines, drawn from the bisections of the three sides of a triangle to the opposite angles, meet in the same point.

(VIII.)

To find, within a given rectilineal angle, the *locus* of all the points, from each of which, if two straight lines be drawn, so as to meet the lines containing the given angle, and so as always to be parallel to two straight lines given in position, they shall be to one another in a given ratio.

(IX.)

If a circle be touched, in the same point, both externally and internally, by two other circles, and through the point of contact two straight lines be drawn, the parts of them intercepted between the circumference of the given circle, and that of the circle which touches it internally, shall have to one another the same ratio as the parts which are chords of the other circle.

(X.)

From the center of a given circle, to draw a straight line to meet a given tangent to the circle,

so that the segment of the line between the circle and the tangent shall be any required part of the tangent.

(XI.)

From a given triangle to cut off a rhombus; the base of the rhombus being part of the base of the triangle, and having its extremity in a given point of that base.

(XII.)

If two given circles touch each other internally, and a chord of the greater, which is perpendicular to the straight line joining their centers, also touch the less, to describe a circle which shall touch the two given circles, and also touch the chord on the same side as the less circle touches it.

(XIII.)

If two triangles have one angle of the one, equal to one angle of the other, and also another angle of the one, together with another angle of the other, equal to two right angles, the sides about the two remaining angles shall be proportionals.

(XIV.)

If, from the extremities of the base of a given triangle, there be drawn two straight lines, both on the same side of the base, and each equal to the adjacent side, and making with that side an angle

equal to the vertical angle of the triangle, then the straight lines which join the extremities of the lines so drawn, and the further extremities of the base, shall cut off, from the sides, equal segments towards the vertex; and each of those segments shall be a mean proportional between the other segments, that are towards the base.

(XV.)

If at the extremities of the hypotenuse of a right-angled triangle two straight lines be drawn, on the same side of the hypotenuse as the right angle, each equal to, and each perpendicular to, the adjacent side, the two straight lines joining each of their extremities and the further extremity of the hypotenuse, shall cut each other in the same point of the perpendicular drawn to the hypotenuse from the right angle.*

(XVI.)

The semi-circumference of a circle having been divided into any number of equal parts, and chords having been drawn, from either extremity of the diameter, to the several points of division, the first chord has to the second, the same ratio which the second has to the aggregate of the first and third;

* This proposition explains a circumstance belonging to the figure of E. 47. 1; and it may very easily be proved, *ex absurdo*, by the help of the third deduction from E. 32. x. XII. set down in this Appendix.

or the same ratio which any other chord has to the aggregate of the two chords that are next to it.

## Prop. VI.

(XVII.)

If two trapeziums have an angle of the one equal to an angle of the other, and if, also, the sides of the two figures, about each of their angles, be proportionals, the remaining angles of the one shall be equal to the remaining angles of the other.

## Prop. VIII.

(XVIII.)

To divide a given finite straight line into two parts, such, that another given straight line, not greater than the half of the former, shall be a mean proportional between them.

(XIX.)

If two straight lines touch a circle at opposite extremities of its diameter, any other tangent of the circle, terminated by them, is so divided in its point of contact, that the radius of the circle is a mean proportional between its segments.

(XX.)

If two given circles touch each other, and also touch a given straight line, the part of the line between the points of contact, is a mean proportional between the diameters of the circles.

(XXI.)

Two straight lines being given, which are the two first of a series of proportionals, to find the rest; and, if the series decrease, to find a line which shall be greater than the aggregate of any number, whatever, of its terms, but to which the aggregate may approximate indefinitely.

Cor. The first term of a decreasing series of proportionals is a mean between the excess of the first term above the second, and the line which is the limit of all the terms.

## Prop. X.

(XXII.)

To describe a square which shall have a given ratio to a given rectilineal figure.

(XXIII.)

To divide a given finite straight line into two parts, the squares of which shall be to one another in a given ratio.

(XXIV.)

To cut off from a rectangle a similar rectangle which shall be any required part of it.

(XXV.)

To find two points, situated in two adjacent sides of a given oblong, at equal distances from two opposite angles, from which, if two straight lines be drawn parallel to the sides of the figure, they shall cut off from it any part required.

(XXVI.)

Hence, within a given rectangle, to describe another rectangle which shall be any required part of it, and shall have its four sides all equally distant from the four sides of the given rectangle.

(XXVII.)

To divide a given straight line into two parts, such, that the rectangle contained by the whole line and one of its parts, shall have a given ratio to the square of the other part.

(XXVIII.)

The base, the vertical angle, and the ratio of the two sides of a triangle being given, to construct it.

(XXIX.)

One given circle lying within another, to find a point from which, if two tangents be drawn, one to each of the given circles, they shall be to each other in a given ratio.

(XXX.)

A given finite straight line being divided into any two given parts, to divide it again, so that the rectangle contained by the two former given parts shall have a given ratio to the rectangle contained by the two latter parts.

(XXXI.)

To draw a straight line to touch a given arch of a circle, so that being terminated by the semi-diameters, produced, which bound the arch, it shall be divided by the point of contact, into two parts that are to one another in a given ratio.

(XXXII.)

Two points being given, one in each of two parallel straight lines, and a third point being also given, without them, to draw, from that third point, a straight line so to cut the parallels, as that the segments of the parallels, between it and the two first points, shall be to one another in a given ratio.

(XXXIII.)

To find a point within a given triangle, from which if three straight lines be drawn to the three angles of the triangle, it shall thereby be divided into three parts that are each to each in given ratios.

(XXXIV.)

The base, the perpendicular distance of the vertex from the base, and the ratio of the two sides of a triangle being given, to construct it.

(XXXV.)

To divide a given circular arch into two parts, so that the chords of those parts shall be to each other in a given ratio.

(XXXVI.)

To inscribe a square in a given trapezium, which has the two sides about any angle equal to one another, and the two sides about the opposite angle also equal to one another.

(XXXVII.)

To inscribe a square in a given trapezium.

## Prop. XI.

(XXXVIII.)

To determine the locus of the vertices of all the triangles which can be described on a given base,

so that each of them shall have its two sides in a given ratio.

(XXXIX.)

Hence, to find a point, from which if three straight lines be drawn to three given points, they shall be each to each in given ratios.

(XL.)

Hence, a straight line being divided into three given parts, to find a point without it, at which the three parts shall subtend equal angles.

(XLI.)

To find a point in a straight line, given in position, from which, if two straight lines be drawn to two given points, both on the same side of the given line, they shall be to each other in a given ratio.

(XLII.)

In a given parallelogram to inscribe a parallelogram that shall have it's two adjacent sides in a given ratio to one another, and that shall be the half of the given parallelogram.

## Prop. XII.

(XLIII.)

Through the bisection of the base of a given triangle, to draw a straight line cutting the sides,

of which one is produced, so that the segments of the line, between the bisection of the base and the two sides, shall be to one another in a given ratio.

(XLIV.)

From a given point, without a given rectilineal angle, to draw a straight line cutting the two lines which contain the angle, so that the distances of the two intersections from the given point, shall be to one another in a given ratio.

(XLV.)

From a given point, without a given rectilineal angle, to draw a straight line cutting off from the lines which contain the angles, segments, towards the summit of the angle, which shall be to one another in a given ratio.

(XLVI.)

From a given point, to draw a straight line to cut a given circle, so that the distances of the two intersections from the given point, shall be to each other in a given ratio.

(XLVII.)

To find, between two given parallel straight lines, the locus of all the points, from each of which if two straight lines be drawn to the two given parallels, so as always to make with them, towards

the same parts, given angles, they shall be to one another in a given ratio.

(XLVIII.)

Two given circles lying wholly without one another, through a given point, which is between the two circles, and which is posited in the straight line joining their centers, to draw a straight line that shall be terminated by the convex circumferences, and divided, by the given point, into two parts, that are to one another in a given ratio.

(XLIX.)

To find a point, from which if three straight lines be drawn to meet as many given straight lines, so as to make, each with the line on which it falls, an angle equal to a given angle, the lines so drawn shall be, each to each, in given ratios.

## Prop. XV.

(L.)

To make an isosceles triangle, which shall be equal to a scalene triangle, and shall also have an equal vertical angle with it.

## Prop. XVI.

(LI.)

If a straight line, drawn from the vertex of an isosceles triangle cutting the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base, shall be equal to the square of either of the equal sides of the triangle.

(LII.)

Of four straight lines which are continual proportionals, the two extremes being given, and also a line which is equal to the difference of the other two, to find those two lines.

(LIII.)

The semi-aggregate of two straight lines, and also another straight line, which is a mean proportional between them, being given, to find the two lines.

(LIV.)

To make a triangle, which shall have its two sides equal to two given straight lines, each to each, and shall have its base equal to the perpendicular distance of the vertex from the base.

(LV.)

If from any point in the diameter, or the diameter produced, of a given parallelogram, perpendiculars be let fall on the two adjacent sides, produced, if necessary, which meet the diameter, the perpendiculars shall be reciprocally proportional to the sides on which they fall.

(LVI.)

Through a given point, within a triangle, to draw a straight line to meet the sides, either of them being produced, if necessary, so that the rectangle contained by the segments, into which the line is divided by the given point, may be equal to a given square.

(LVII.)

To find a point, from which if three straight lines be drawn to three given points, their differences shall be severally equal to three given straight lines; the difference of any two of the straight lines to be drawn, not being greater than the distance of the two points to which they are to be drawn.

(LVIII.)

To describe a circle, which shall pass through a given point, and touch two given circles.

(LIX.)

To describe a circle that shall touch three given circles.

## Prop. XVIII.

(LX.)

Upon a given finite straight line, to describe an equilateral and equiangular polygon, having the number of its sides equal to four, eight, sixteen, &c.; or to three, six, twelve, &c.; or to five, ten, twenty, &c.; or to fifteen, thirty, sixty, &c. sides.

## Prop. XIX.

(LXI.)

To cut off from a given triangle any part required, by a straight line drawn parallel to a given straight line.

## Prop. XX.

(LXII.)

To describe a polygon, similar to a given polygon, and having a given ratio to it.

(LXIII.)

Any regular polygon, inscribed in a circle, is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

(LXIV.)

If from two points similarly situated, one in each of any two homologous sides of two similar polygons, two straight lines be drawn making equal angles with those sides, they shall cut off from the polygons two similar figures; and the one shall be the same part of the one polygon, that the other is of the other.

## Prop. XXII.

(LXV.)

If any two chords of a circle intersect each other, the straight lines joining their extremities shall cut off equal segments from the chord which passes through the common intersection of the two former chords and is there bisected.

(LXVI.)

Two similar rectilineal figures being given, to find a third figure also similar to them and a mean proportional between them.

## Prop. XXIII.

### (LXVII.)

Equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each.

Cor. Triangles, having equal vertical angles, are to one another as the rectangles contained by the sides about those equal angles.

### (LXVIII.)

Through a given point, either without or within a given triangle, to draw a straight line which shall cut off from the triangle any part required.

Cor. Hence, and by the help of Trigonometry, any given rectilineal figure may be divided into two parts, which are to each other in any given ratio, by a straight line drawn from a given point, situate without the given figure.

### (LXIX.)

If two sides of a trapezium be parallel, and a straight line be drawn cutting them, and meeting also the other two sides, (any of the sides being produced, if necessary) the two rectangles contained by the respective segments of the parallel sides, have to each other the same ratio, as the

two rectangles contained by the segments into which the line, so drawn, is severally divided by each of the two parallels.

## Prop. XXX.

(LXX.)

A given straight line being cut in extreme and mean ratio, if from the greater segment the less be taken, the greater segment also will thus be cut in extreme and mean ratio; and if a straight line, equal to the greater segment, be added to the given line, the line which is made up of the given line and this segment, is also cut in extreme and mean ratio.

(LXXI.)

Upon a given straight line as an hypotenuse, to describe a right-angled triangle, which shall have its three sides continual proportionals.

(LXXII.)

The perimeter being given of a right-angled triangle, having its three sides proportionals, to construct the triangle.

(LXXIII.)

The radius of a given circle having been divided in extreme and mean ratio, the greater segment

shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle.

## Prop. XXXI.

(LXXIV.)

Any rectangle is the half of the rectangle contained by the diameters of the squares of its two sides.

## Prop. XXXIII.

(LXXV.)

In different circles the semi-diameters which bound equal sectors contain angles reciprocally proportional to their circles; and conversely.

(LXXVI.)

To trisect a given circle, by dividing it into three equal sectors.

(LXXVII.)

If, from the greater of two unequal sides, of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.

(LXXVIII.)

The greater of any two unequal arches, of a given circle, has a greater ratio to the less arch, than the chord of the greater has to the chord of the less.

Cor. The greater angle, at the base of a scalene triangle, has a greater ratio to the less angle, than the greater side has to the less side.

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Prop. D.—*Simson's Euclid.*

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(LXXIX.)

If, from the center of the circle, described about a given triangle, perpendiculars be drawn to the three sides, their aggregate shall be equal to the radius of the circumscribed circle, together with the radius of the circle inscribed in the given triangle.

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*In the investigation of the six next following deductions, it is necessary to quote the theorem, which is the Second Proposition of the Twelfth Book of Euclid's Elements.*

(LXXX.)

To divide a given circle into any required number of equal parts, by circles described within it, about its center.

(LXXXI.)

To find a circle, which shall be equal to the excess of the greater of two given circles above the less.

(LXXXII.)

If, in any given circle, two chords cut each other at right angles, the four circles described upon their segments, as diameters, shall, together, be equal to the given circle.

(LXXXIII.)

A circle is a mean proportional between any regular polygon, described about it, and a similar polygon, the perimeter of which is equal to the circumference of the circle.

(LXXXIV.)

If a figure be bounded by two circular arches, subtending at their respective centers angles reciprocally proportional to the circles to which they belong, a square may be found, that shall be equal to it.

(LXXXV.)

A circle is equal to the half of the rectangle contained by its circumference and its semidiameter.

Cor. The circumferences of circles are to one another as their semi-diameters.

*The following Propositions were omitted in their proper places:*

## BOOK I.

### Prop. XXXIV.

(XLII. A.)

If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

### Prop. XXXVIII.

(LVI. A.)

If two triangles have the two adjacent sides of a parallelogram for their bases, and have their common vertex situated in the diameter, or in the diameter produced, they shall be equal to one another.

## BOOK III.

### Prop. XVI.

(VII. A.)

The diameter of a circle having been produced to a given point, to find in the part produced, a

point from which if a tangent be drawn to the circle, it shall be equal to the segment of the part produced, that is between the given point and the point found.

## PROP. XXXVI.

(LXXIV. A.)

To find a point from which if straight lines be drawn to touch three given circles, none of which lies within another, the tangents so drawn shall be equal to one another.

## PROP. XXXVII.

(LXXXV. A.)

To describe a circle which shall have its center in a given straight line, which shall pass through a given point, and shall, also, touch another given straight line.

# APPENDIX II.

CONTAINING

## A SERIES OF PROPOSITIONS

WHICH

MAY BE SOLVED AND DEMONSTRATED

BY

THE PRINCIPLES

OF

**Natural Philosophy**

AND

**Geometrical Constructions.**

# APPENDIX II.

## PROPOSITIONS

IN

## NATURAL PHILOSOPHY.

(I.)

To determine the directions in which two forces, represented, in quantity, each by a given straight line, must act, so that the equivalent compound force may be represented by another given straight line; the two former straight lines being, together, greater than the latter.

(II.)

To resolve a given force into two forces, of which the directions shall make with one another an angle equal to a given angle, and the quantities shall, together, be equal to the quantity of another given force.

(III.)

To resolve a given force into two forces, of which the directions shall make with one another

an angle equal to a given angle, and which shall have the difference of their quantities equal to another given force.

(IV.)

To resolve a given force into two forces, the quantities of which shall be to one another in a given ratio, and the directions shall make with each other an angle equal to a given angle.

(V.)

There being given the quantity and the direction of one of the two forces which act upon a body, the direction of the other force, and also the force which is equivalent to them both, to find the quantity of the other force.

(VI.)

Three points *A, B, C,* being given upon any plane, it is required to determine a point, *P*, on the same plane, to which if straight lines be drawn from *A*, *B*, and *C*, they shall be the directions in which three forces, each given in quantity, must act upon a body at *P*, so as to keep it at rest.

(VII.)

The quantity and the direction being given of a force, which acts upon a body at rest, to determine the quantities and the directions of three equal forces, which also acting upon the body,

each in a direction at right angles to the directions of the other two, shall keep it at rest.

(VIII.)

The quantities and the directions being given, of two forces which act upon a given straight lever, at the extremities of its arms, and thereby keep the lever at rest on a fulcrum, to determine, geometrically, the place of the fulcrum.

(IX.)

The place of the fulcrum in a straight lever being given, and the quantities of two forces which are to act at its extremities, to determine the directions in which the two forces must act, so that these forces and the pressure on the fulcrum may be to each other as three given straight lines, of which any two are greater than the third.

(X.)

Two forces, each given in quantity, are to act upon a straight lever, of indefinite length, at two given points, both on the same side of the fulcrum: It is required to determine the place of the fulcrum, and the directions of the given forces, so that the pressure on the fulcrum may be to each of the given forces, as each of two given straight lines to a third given straight line, and may take place

in a direction which shall make with the lever an angle equal to a given angle.*

(XI.)

The force of gravity being supposed to tend to the earth's center, but to be a constant force, the pressure on the fulcrum of a straight lever, at the ends of which two given weights are appended, is the less, the nearer the lever is to the earth's center.

(XII.)

To determine the position of a given straight lever, loaded at a given point with a given weight, so that it may rest when placed between two planes, each of them inclined to the horizon at a given angle.

(XIII.)

The arms of a bent lever being given, in length, and the angle being given, which they make at the fulcrum, to determine the *position* of the lever, when two given weights, appended at its extremities, balance one another.

(XIV.)

A flexible string, loaded at one of its ends with a given weight, and passing over a fixt pulley, has

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* Any two, of the given straight lines, are supposed to be greater than the third.

its other end fastened to a given point in the same vertical plane with the pulley, and it is further loaded, at a given point between the pulley and the fastened end of the string, with another given weight: It is required to determine any number of positions of this latter weight, in each of which it will balance the former weight.

(XV.)

Three given weights being affixed to the same string, and the two extreme weights being made to act upon the middle weight, by means of two fixt pullies, which are in the same vertical plane, to determine the position of the string, when the whole system of weights is at rest.

(XVI.)

The position being given of a string, loaded with two weights, at two given points, and having its ends fastened at two given points in its vertical plane, it is required to find two straight lines, which shall be to one another as the two weights.

(XVII.)

A string of given length being supposed to be fastened at two given points in a vertical plane; to find the point in the string at which a given ring will rest.

(XVIII.)

A system of given spheres, of equal weights and magnitudes, being so placed, as that the two

extreme spheres resting on two inclined planes, the whole system is supported in the form of an arch, it is required to determine the places of the centers of the spheres, and the position of the two inclined planes.

(XIX.)

To find the center of gravity of a trapezium, which has two of its sides parallel to one another.

(XX.)

If the sides of a given triangle, taken in order, be cut proportionally, and the points of section be joined; the center of gravity of the triangle so formed will be the same as that of the given triangle.

(XXI.)

The centers of gravity of all the parallelograms which can be inscribed in a given parallelogram, will be in the same point as the center of gravity of the given parallelogram: And the center of a circle is the common center of gravity of all the rectangles that can be inscribed in it.

(XXII.)

If a given body remain at rest either within or without a given circle, or in its circumference, and another given body describe the circumference; to find the path of their center of gravity.

(XXIII.)

If any number of equal bodies be placed in the circumference of a given circle, to determine the locus of the several centers of gravity of any one of them, and each of the rest.

(XXIV.)

To find the path of the common center of gravity of two bodies, which move uniformly through two sides of a given triangle, in the same time, and are to one another inversely as those two sides.

(XXV.)

The paths of two bodies, which move with known uniform velocities, being given, so that their places at any given instant are known, it is required to determine the relative path, and the relative velocity, of the one, as seen from the other.

(XXVI.)

The position being given of two perfectly elastic balls on a table, the sides of which constitute a given polygon, to find the direction in which the one ball must be sent, so that after impinging successively on each of the sides, it may at last strike the other ball.

(XXVII.)

The positions being given of two perfectly elastic balls (*A*) and (*B*), on an horizontal table, which is

in the form of a rectangular parallelogram, to find the direction in which ($A$) must be sent, so that impinging (1) on one side, (2) on two of the sides, (3) on three of the sides, or (4) on four of the sides, it may at last strike the other ball.

(XXVIII.)

Given the position of a perfectly elastic ball ($A$), on an horizontal plane, to find the locus of the points in which another given elastic ball ($B$) can be placed, so that after ($A$) impinges upon ($B$), ($A$) and ($B$) may strike two given points respectively.

(XXIX.)

Two planes, which rest upon the level ground, so as to have their common section perpendicular to the horizon, are inclined to one another at a given angle, and a perfectly elastic ball, sent along the ground, so as always to strike a given point in the one, is reflected towards the other: It is required to determine the locus of the points, on the ground, from which if the ball be sent, it shall, in each case, after the two reflections, return to its point of projection.

(XXX.)

If the secant of a circular arch be in a vertical position, the times in which a heavy body, falling from a state of rest, would describe the secant, the

tangent, and the radius drawn through the extremity of the arch, are all equal.

(XXXI.)

If any number of heavy bodies be let fall, at the same instant, the one from the upper extremity of the vertical diameter of a given circle, the rest from the several upper extremities of chords drawn to the lower end of that diameter, so as to describe the diameter and the chords respectively, it is required to find the locus of the positions of the bodies, at any given time of their descent.

(XXXII.)

To determine the position of a straight line to be drawn from a given point to a given inclined plane, so that the time in which a heavy body falls from rest, down that line, may be equal to the time in which it would fall down the given plane.

(XXXIII.)

From a given point in an horizontal plane, to draw a straight line to a given perpendicular to the plane, which shall be described by a heavy body, falling from rest, in the same time as the given perpendicular would be described.

(XXXIV.)

From the right angle of a given right-angled triangle, having one of the sides containing the

right angle parallel to the horizon, to draw to the hypotenuse a straight line, such, that the times of descent down the line so drawn, and down the hypotenuse, shall be equal.

(XXXV.)

To place between an horizontal straight line of indefinite length, and a given perpendicular to it, a straight line of given length, less than the given perpendicular, which shall be described by a heavy body, falling down it from a state of rest, in the same time as the perpendicular.

(XXXVI.)

To place between an indefinite horizontal straight line and a given perpendicular to it, a straight line which shall make with the horizon an angle equal to a given angle, and shall be described by a heavy body in the same time as the given perpendicular.

(XXXVII.)

In a given circle, the plane of which is vertical, to draw a diameter which shall be described by a heavy body, in any given time, greater than the time in which the vertical diameter is described.

(XXXVIII.)

If, from any point in the circumference of a vertical circle, straight lines be drawn to the extre-

mities of the vertical diameter, and if these lines be cut by a parallel to that diameter, the times of descent down the two segments, which are towards the diameter, shall be equal.

(XXXIX.)

If two straight lines, drawn from a given point of an horizontal plane, to a given vertical straight line, be equidistant from the straight line which makes with the horizon, at the given point, an angle equal to half a right angle, they shall be described, by falling bodies in equal times.

(XL.)

The time in which a heavy body, moving with the velocity acquired in falling down a given inclined plane, describes a given horizontal space, is to the time of its falling down the plane, as the half of the given space so described, is to the length of the inclined plane.

(XLI.)

If the plane of a scalene triangle be vertical, and the greater side exceed the less by the half of the base, the time of falling down the greater side is equal to the time of falling down the less, together with the time of describing the base, with the velocity thence acquired.

(XLII.)

The base and one side being given in length, of a scalene triangle, the plane of which is vertical, to construct the triangle, so that the time of falling down the greater side may be equal to the time of falling down the less, together with the time of describing the base, with the velocity thence acquired.

(XLIII.)

The plane of a scalene triangle being vertical, and one side and the angle which it makes with the base being given, to construct the triangle, so that it shall have the same property as that described in the last problem.

(XLIV.)

If the plane of a scalene triangle be vertical, the time of falling down the greater side shall be equal to the time of falling down the less, together with the time of describing, with the velocity thence acquired, a part of the base, that is equal to the double of the excess of the greater side above the less.

(XLV.)

If two bodies, after falling from the summit of a vertical scalene triangle, down the sides, begin

to move along the base, each with its acquired velocity, to determine the point in which they will meet.

(XLVI.)

To find a point, in an horizontal plane, such, that if two straight lines, of given unequal lengths, be drawn from it to a given vertical straight line, the times of falling down them shall be equal.

(XLVII.)

The tangent of a circular arch being horizontal, if a body, after falling down the vertical radius to the center, move along the secant with its acquired velocity, the space which it will describe, in the same time as that of its fall, shall be equal to the radius, together with the perpendicular distance of the point, in which the secant cuts the circumference, from the tangent.

(XLVIII.)

If two heavy bodies let fall, at the same time, from two given points in a vertical straight line, move, after reaching the ground, with their acquired velocities, along the same horizontal straight line, to determine the point at which the one will overtake the other.

(XLIX.)

Two given parallel straight lines, of unequal lengths, being equally inclined to the ground plane

in which they terminate, and being at a given distance from each other, if two heavy bodies, let fall at the same instant from their summits, move after reaching the ground, with their acquired velocities, along the same horizontal straight line, to determine the point in which the one will overtake the other.

(L.)

The *data* being the same as in the last problem, excepting that the two given straight lines, although equally inclined to the horizon, are not parallel to one another, to find where the two bodies will meet.

(LI.)

To place, on the ground, two given unequal vertical straight lines, at such a distance asunder, as that the time of falling down the greater, and then moving, with the velocity acquired, to the less, shall be equal to the time of falling down the less, and then moving, with the velocity thence acquired, to the greater.

(LII.)

The plane of a given right-angled triangle being vertical, and one of the sides containing the right angle being parallel to the horizon, and being taken as the base, to find a point in the perpendicular, produced, if necessary, such that the time of falling from it to the base, and afterwards describing the

base, with the acquired velocity, shall be equal to the time of falling down the perpendicular and then describing the base, with the velocity so acquired.

(LIII.)

If the vertical diameters, of two vertical circles, be in the same straight line, and have a common lower extremity, from which several straight lines are drawn cutting both the circumferences, the times in which heavy bodies fall down the segments of these lines, between the two circumferences, are all equal.

(LIV.)

If the plane of a circle be vertical, and a tangent be drawn to it at the upper extremity of its vertical diameter, the time in which a heavy body falls, from any point in the tangent, to the convex circumference, and afterwards with its acquired velocity, describes a chord drawn parallel to the tangent, from the point in which the body strikes the convex circumference, is equal to the time of falling from the same point to the concave circumference, and of afterwards describing, in like manner, the chord drawn parallel to the tangent, from the point in which the concave circumference is struck.

(LV.)

Of all inclined planes having a common base, to determine that which shall be described by a falling body in the least time.

(LVI.)

Two sides of a triangle being given, the plane of which is vertical, to construct it, so that the time of falling down the third side may be a minimum.

(LVII.)

Two points, equidistant from an horizontal plane, being given, to find a point in the plane, such, that if straight lines be drawn from it to the two given points, the time in which a heavy body falls down the one and then ascends the other, may be a minimum.

(LVIII.)

To find the straight line of quickest descent to a given plane, from a given point above it.

(LIX.)

To find the straight line of quickest descent from a given plane, to a given point below it.

(LX.)

To find the straight lines of quickest descent to the circumference of a given vertical circle, from a given point, either within or without the circle, but in the same plane with it.

(LXI.)

To find the straight lines of slowest descent, (1) to the convex, (2) to the concave circumference of a given circle, from a given point above, and in the same plane with, the circle.

(LXII.)

Two given circles being in the same vertical plane, to find the straight lines of quickest and of slowest descent, drawn from the circumference of the one to the circumference of the other.

(LXIII.)

If a body descend from the highest point of a vertical cycloid of which the base is parallel to the horizon, and the vertex downwards, the time of falling down any arch of the cycloid varies as the arch of the generating circle, intercepted between the base of the cycloid, and a parallel to the base drawn from the place of the body at the end of that time.

(LXIV.)

If any number of bodies fall down different vertical cycloids, that have their bases in the same parallel to the horizon, and terminated at one extremity in the same point, from which the bodies begin their motion, at the same instant, each of them will arrive, sooner than the rest, at that per-

pendicular to the horizon, which is the axis of its curve.

(LXV.)

A given point on the ground, and two given points in the opposite sides of a vertical rectangle being all in the same vertical plane, it is required to find the direction and the velocity of projection, with which a ball, sent from the given point on the ground, shall pass through the other two given points.

(LXVI.)

The velocity of projection being given, to find the direction in which a body must be thrown, so that the aggregate of the altitude and amplitude shall be a maximum.

(LXVII.)

If a body be projected from a given point on the ground, with a given velocity, in a direction making with the horizon an angle equal to half a right angle, and if another body be thrown horizontally, with an equal velocity, from a point directly above the given point, and at a distance from it equal to the space due to the common velocity of projection, the two bodies so thrown will strike the ground in the same point.

(LXVIII.)

The velocity with which a ball is shot from a cannon at a mark on the ground, and also the velo-

city of sound, being given, to find a point in the perpendicular drawn, from the mark, to the straight line joining the place of the cannon and the mark, at which the explosion of the cannon, and the blow upon the mark, will be heard at the same time.

(LXIX.)

If any number of bodies be projected from the same given point, with equal and given velocities, the focus of the parabola described by any one of them shall be in the surface of a sphere, having the point of projection for its center, and the space due to the common velocity for its radius: And the vertex of the parabola described by any of the bodies shall be in the surface of a spheroid, of which the major-axis is horizontal, and is the double of the minor-axis.

(LXX.)

If any number of projectiles be thrown from a given point, at the same instant, and with equal velocities, at any given time, they shall all of them be found in the circumference of some circle, the center of which is in the vertical straight line drawn through the given point of projection.

(LXXI.)

To find a point in a given horizontal plane, and the direction in which a body must be projected from that point, with a given velocity, so that a

given point above the plane may be the highest point of the trajectory described.

(LXXII.)

To find the velocity with which a perfectly elastic ball must be projected up a given inclined plane, so that, being reflected at the top by a vertical plane, it may come to the hand again.

(LXXIII.)

To find the locus of all the points from which, if a perfectly elastic ball be successively thrown, with a given velocity, so as always to strike a given vertical plane in the same given point, it shall in each case return to the point from which it was projected.

(LXXIV.)

To find the least velocity with which a body, projected from the top of a given inclined plane, so as to describe a parabola, shall strike the bottom of the plane.

(LXXV.)

To determine the furthest point, in a given horizontal plane, which can be hit by a body projected with a given velocity, from a given point without the plane.

(LXXVI.)

To find the locus of the foci of all the parabolas described by perfectly elastic balls, let fall from a given horizontal line, upon a given inclined plane.

(LXXVII.)

The same supposition being made as in the preceding problem, to find the point in the horizontal line, from which, if a perfectly elastic ball be dropped upon the inclined plane, the range shall be a maximum.

(LXXVIII.)

To determine the highest and lowest points in a given vertical plane, which can be hit by a body projected, with a given velocity, from a given point without the plane.

(LXXIX.)

To determine the highest and lowest points in a given inclined plane, which can be hit by a body projected, with a given velocity, from a given point without the plane.

(LXXX.)

The base of an inclined plane being given, to find its altitude, so that a ball being projected up the plane with a given velocity, may, after leaving the plane, strike the base produced in a given point.

(LXXXI.)

If a perfectly elastic ball projected from an horizontal plane, perpendicularly upwards, with a given velocity, impinge upon a perfectly hard and immoveable plane, inclined to the horizon, at a

given angle, to find where the ball will again strike the horizontal plane.

(LXXXII.)

A perfectly elastic ball being supposed to be let fall from a given point, above an horizontal plane, to find where a given inclined plane must be placed, so that the ball, after impinging upon it, may strike the horizontal plane in a given point.

(LXXXIII.)

If any number of given inclined planes have a common section which is parallel to the horizon, to determine the locus of the furthest points of the planes, which can be hit, by a body projected from a given point in the common section, with a given velocity.

(LXXXIV.)

If any number of bodies, projected with equal velocities, from the same given point, in different horizontal directions, all strike a given plane, which is situated below the given point, to determine the locus of the points of the plane that are hit; (1) if the given plane be horizontal; (2) if the given plane be vertical; and (3) if the given plane be inclined to the horizon.

(LXXXV.)

To determine the graduation of a given glass cylinder, so that the several numbers of degrees may

indicate the specific gravities of different fluids, in which the cylinder floats, the specific gravity of pure water being represented by unity.

(LXXXVI.)

A cylinder filled with water being given, the altitude of which is ($a$) inches, to divide it into ($n$) parts, on which the lateral pressures shall be equal.

(LXXXVII.)

The place of the lateral orifice, in a close cylindrical vessel, containing water, being given, and also the point of the ground struck by the issuing fluid, to find the altitude of the water in the vessel.

(LXXXVIII.)

The direction of a tube, inserted at a given point in the side of a close cylindrical vessel, containing water, being given, and also the point of the ground struck by the spouting fluid, to determine the altitude of the water in the vessel, and the track of the spouting fluid.

(LXXXIX.)

To find the place of an orifice to be made in the side of a given cylinder, filled with water, and inclined to the horizon at a given angle, so that the effluent fluid may just strike the base of the cylinder.

(XC.)

To find the place of an orifice to be made in the side of a given cylinder filled with water, and inclined to the horizon at a given angle, so that the fluid may spout the farthest from its base on an horizontal plane.

(XCI.)

If a hollow sphere elevated above the ground and filled with the water, be supposed to be bored through, in every point of its surface, to find a concave surface, which shall be touched by the several streams of the spouting fluid.

(XCII.)

A prismatic vessel has, at a given point in its side, a circular orifice of given dimensions, and is kept full by a certain supply of water: It is required to find the dimensions of another such an orifice to be made at another given lower point in the side, so that the same supply of water may keep the vessel full, when this lower orifice is open, and the upper orifice closed.

(XCIII.)

A vessel, from which the water issues, at the bottom, is supplied at a given rate with water, poured into its top, and the surface is thus kept at a certain known altitude: It is required to find the

proportionate additional supply of water, with which the vessel must be fed, so that the surface of the fluid may be kept at any greater given altitude.

(XCIV.)

In the straight line joining the extremities of two given finite straight lines, to find a point from which the two given lines being seen, they shall be of the same apparent magnitude.

(XCV.)

In a straight line, parallel to the straight line which joins the extremities of two given finite straight lines toward the same parts, to find a point, from which the two given lines being seen, they shall be of the same apparent magnitude.

(XCVI.)

To find a point, from which two sides of a given triangle being seen, they shall be of the same *given* apparent magnitude.

(XCVII.)

The vertical angle of a triangle being less than the exterior angle of an equilateral triangle, to find within it a point, from which if the three sides be seen they shall be of the same apparent magnitude.

(XCVIII.)

In a given indefinite straight line, to find a point, seen from which a given finite straight line,

lying wholly without the indefinite line, shall be of the greatest apparent magnitude.

(XCIX.)

To find the locus of all the points, in which an eye can be placed, so that two unequal straight lines, both lying in the same straight line, may always appear of the same magnitude.

(C.)

To find the point, in which an eye must be placed, so that three given straight lines, all situated in the same straight line, may appear of the same magnitude.

(CI.)

Two points being given, one on each side of a given indefinite straight line, to find a portion of the indefinite line, such that its apparent magnitude, when seen from one of the given points, shall be the double of its apparent magnitude, when seen from the other.

(CII.)

To find a point in the circumference of a circle, seen from which two given unequal portions of a given straight line, without the circle, shall be of equal apparent magnitudes.

(CIII.)

If a plane mirror revolve about an axis, the angular motion of the image of any object is the double of the angular motion of the mirror.

(CIV.)

A given straight line being inclined to the horizon at a given angle, to find the position of a plane mirror, from which the image of the given straight line shall be inclined to the horizon at any other given angle.

(CV.)

A given vertical object being supposed to be at a given height above an horizontal plane, to find the point in the plane, where the apparent magnitude of the object is greatest.

(CVI.)

The eye being supposed to be placed in any point of a given parallelogram, to determine the least length and breadth of a parallel plane mirror, in which the whole of the image of the given figure shall be visible.

(CVII.)

A circle which is inclined at a given angle to the horizon, is placed opposite to a plane vertical

mirror: it is required to determine the greatest length and breadth of the least portion of the mirror, in which the whole image of the circle shall be visible to an eye placed in its center.

(CVIII.)

The eye being in the bisection of a given straight line, which is parallel to a vertical mirror, to draw through the place of the eye another straight line, of a given length, so that its image shall be of the same apparent magnitude as the image of the first-mentioned given straight line.

(CIX.)

The eye and a given luminous point being situated between two plane mirrors, inclined to one another, the eye's distance from any one of the images of the given point is equal to the aggregate of the incident ray and the reflected rays, belonging to that image.

(CX.)

A radiant point, and the position of the eye, being given, both on the same side of a plane mirror, and the mirror being supposed to move in a direction perpendicular to its own plane, to find the locus of the several points of the mirror, from which rays are reflected to the eye.

(CXI.)

To find an incident ray, parallel to the axis of a given reflecting circular arch, which shall be reflected so as to pass through a given point in the axis.

(CXII.)

To find an incident ray, parallel to the axis of a given reflecting circular arch, that shall be reflected so as to pass through a point in the circumference, which is at the distance of half a quadrant from the intersection of the axis and the mirror.

(CXIII.)

If two rays fall on the convex side of a reflecting circular arch, parallel to its axis, the arch of the mirror intercepted between the two incident rays, is one-third of the arch contained between the directions of the reflected rays, on the contrary side of the point in which they cross one another.

(CXIV.)

A reflecting circular arch, and two points in the same plane with it, being given, to find a ray proceeding from the one, supposed to be a luminous point, which shall be reflected in a direction passing through the other; (1) when both points are in the circumference; (2) when only one of the points is in the circumference; (3) when both points are equidistant from the center, but are not

in the circumference; (4) when both points are in the axis of the reflector.

(CXV.)

Two points, one of which is a radiant point, being given, both situated in the same perpendicular to a refracting surface, and both on the same side of the surface, to find a ray proceeding from the one, which shall be refracted in a direction passing through the other; the ratio of the sine of incidence to the sine of refraction being also given.

(CXVI.)

If a luminous point describe a circle, having its center in the axis of a double convex, or a double concave lens, and its plane perpendicular to the axis, the image shall describe another circle, having its plane also perpendicular to the axis.

(CXVII.)

To find a point in the axis of a given glass lens, such that when it is the focus of incident rays, the focus of emergent rays shall be at an equal distance from the center of the lens.

(CXVIII.)

To find a sphere, such that if a luminous point, placed before a double convex lens, move in any track over part of its surface, the image shall

describe a similar track, and shall move over another part of its surface.

(CXIX.)

One of the surfaces of a given glass lens being coated with a reflecting substance,* to find a point in the axis of the lens, such that all rays which fall upon the lens diverging from it, or converging to it, shall pass through it again, after being once reflected, and twice refracted, by the lens.

(CXX.)

To find the focus of a small pencil of parallel rays, falling upon a given glass lens-mirror, of inconsiderable thickness.

(CXXI.)

If one side of a *meniscus*, bounded by concentric spherical surfaces, be coated with a reflecting substance, the distance of the principal focus of rays, will be the same as in the case of simple reflexion from the spherical mirror.

(CXXII.)

The radius of the refracting surface of a glass lens-mirror, of inconsiderable thickness, being

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* A lens so prepared, is called, in two of the following propositions, a *lens-mirror*.

given, to find the radius of the reflecting surface, so that when a very small pencil rays proceeds from a given point in the axis, the directions of the rays, after one reflection, and two refractions, shall again pass through the given point.

(CXXIII.)

If a ray parallel to the axis of a glass sphere, one half of which is covered with a reflecting substance, fall upon the refracting half of the sphere, after refraction it shall be reflected to a point in the axis, at a distance from the reflecting surface nearly equal to one-sixth part of the sphere's diameter.

(CXXIV.)

The orbits of the earth and a planet being supposed to be concentric circles, and to be in the same plane, to find the point in which the planet, seen from the earth, appears to be stationary.

(CXXV.)

To determine the angle which, at any given time, the straight line joining the moon's cusps will make with the horizon; the moon's orbit being supposed to coincide with the ecliptic.

(CXXVI.)

According to the Simple Elliptic Hypothesis, the earth being situated in one focus of the moon's

orbit, the plane of the same lunar meridian will continually pass through the other focus.

(CXXVII.)

The earth being supposed to be a sphere, and the angles being observed, which the straight line, joining the tops of two distant mountains, makes with the plumb-line, at each of the tops, to determine the dimensions of the earth.

(CXXVIII.)

The time of the beginning of morning twilight, or of the end of evening twilight, being given, to find the altitude of the atmosphere, which reflects the sun's light.

(CXXIX.)

The annual parallax of a fixt star being supposed to be sensible, to determine the points of the earth's orbit in which the annual parallax in latitude, and the annual parallax in longitude, are the greatest.

(CXXX.)

The earth being supposed to move with an uniform velocity, in a circular orbit, and the sun being supposed to be placed in a given point, which

*k*

is not the center of the orbit, to determine in what points of the orbit the difference of the angles, described about the center of the circular orbit and the center of the sun, is a maximum; the angles being measured from the apogee.

ON

# MAXIMA AND MINIMA.

## INTRODUCTION.

Geometry, like every other system of useful knowledge, is said to have had its rise in the actual necessity of inventing it. What is wanting in authentic history on this point, is compensated by the highest degree of probability. Not only does the name itself indicate the occasion upon which it was given, but the natural features of the particular region, which has been assigned as the nursery of this science, are such as to give credibility to the traditionary account of it. In a country where the quantity of cultivated soil is necessarily small, where the population has always been numerous, where fertility is placed in immediate contrast with the extreme of barrenness, and where the boundaries of property are liable to be



16

swept away by an annual inundation, the practice and the theory of the correct mensuration of plane surfaces could hardly fail to originate. But if such be the origin of Geometry, that particular branch of it, which is the subject of the first of the following Sections, was not, perhaps, unknown to the ancient Egyptians. Where there was a choice of figure, some consideration is likely to have been bestowed on the most œconomical plan of division, or enclosure: and many of the mathematical truths which belong to this subject are so obvious, that they could not well escape even a cursory survey. If a field, in the form of a rectangular parallelogram, were to be divided equally between two persons, it might be done either by drawing the diagonal, or by drawing a straight line perpendicular to one of the sides through the point of its bisection. Which of these two methods of bipartition would require the least fencing is sufficiently evident. The advantage of the square, in this respect, compared with an equal triangle or parallelogram, and some other elementary truths of the same kind, may also be supposed to have been early known. The error arising, in common cases, from considering the earth's surface as plane, instead of treating it as a convex surface, is altogether inconsiderable: Not to mention, that some of the leading propositions, relating to maxima of enclosed space, and to minima of boundary, are found to be the same in Plane and in Spherical Geometry.

It must, however, be allowed, that the full investigation of such maxima and minima would not be called for by the wants of men. The case which would prompt it, the transfer of a learned nation into a country wholly unenclosed, and not abounding, in any part of it, with materials for fencing, has never yet occurred. The mythology of the ancients, indeed, exhibits an instance, which might have afforded full scope for speculations of this kind. But this is fabulous. In long established communities the divisions of land have taken place gradually, and upon no regular plan; nor is there often an opportunity of making any extensive application, in this way, of the principles of science. It is, however, worth knowing, how a given quantity of surface may be most œconomically fenced, in a given manner, and which is the best of all figures in this respect, where there is no restrictions, or the best figure of its kind, if the number of sides be prescribed: it is not an uninteresting problem to determine, what form a given length of outline must be made to assume, in order that it may contain the greatest area; or, what is the best method of subdividing a given space into a given number of equal and similar parts, so as to leave no interstices. We cannot wonder, then, that the immutable relation which exists between the space enclosed and the species of its boundary, should have engaged the attention of the ancient mathematicians.

Apollonius of Perga, who flourished under Ptolemy Euergetes, treated largely of Maxima

and Minima, in the fifth book of his work on Conic Sections. This work of the "Great Geometrician," consisting of eight books, remained entire until the fourth century, when it was in the hands of Pappus. It appears, indeed, to have been known, in its perfect state, to Eutocius, at the latter end of the fifth century. But, from that time to the year 1658, the first four books only were known to be extant. The discovery of a Compendium of the three following books in Arabic, by Borelli, and the publication of the very ingenious attempt of Viviani to supply from conjecture the matter of the fifth book, bear nearly the same date: but it is very certain that that eminent scholar of Galileo could not have previously seen the Arabic manuscript, which was found by accident in the Medicean Library at Rome.

The Mathematical Collections of Pappus have reached us in a mutilated state. They are now hardly accessible at all, excepting through a Latin translation which is full of errors. This author appears to have been led to the consideration of Geometrical Maxima and Minima by observing the cells of bees: and it is to be regretted that we have not his complete investigation of that most astonishing structure. What he has delivered, in his fifth book, has found its way into the writings of many modern mathematicians.

In the literary correspondence which took place between Torricelli, Fermat, and Roberval, toward the beginning, and the middle, of the seventeenth

century, geometrical questions of this kind seem to have been proposed: there is one, in particular, sent from Fermat to Torricelli, the solution of which will be given in the course of this work.

Dr. Barrow, in the Addenda to his Lectiones Geometricæ, a work of great originality and erudition, has shewn how theorems relating to maxima and minima may be deduced from the consideration of tangents. His two fundamental theorems are demonstrated with admirable perspicuity and elegance.

Amongst the works of later mathematicians, not to mention those of L' Huillier and other foreigners, the Elements of Thomas Simpson contain a series of propositions, "on the Maxima and Minima of Geometrical Quantities," in which there is not much that is original.

Waring, in his Treatise on Curve Lines, has determined several maxima and minima, of which it may be said, that they are curious rather than useful.

But the two last-mentioned authors, as well as Pappus among the ancients, and most of the modern writers upon this subject, have admitted into their reasonings what appears to be a sophism. It is easily shewn, for example, that of all triangles standing upon the same base and of equal perimeter, that which is isosceles is the greatest: hence they have concluded, at once, that whatever the rectilineal figure be, of a given perimeter, when it is greatest, its sides are all equal. Thus they have

taken for granted that the quantity under consideration has a maximum value: which is not allowable in any geometrical proposition. Such an assumption may be properly made in the analytical investigation of maxima and minima; because if no maximum nor minimum exists, the process itself will shew that to be the case, by the nature of its result: and if no absurdity appear in the result, and it still be doubtful whether the quantity determined give a maximum or a minimum value, there are means of ascertaining to which of the two it belongs. But, in this application of Geometry, it is necessary to prove that the variable magnitude is greater, or less, than any other of the same kind with itself, before the conclusion can be fairly drawn.

The first division of the following publication is purely geometrical, and an easy application for the most part of the Elements of Euclid. Wherever any theorem or problem is wanted, which is not contained in that book, it has been supplied: in those cases, in which a proposition relating to maxima and minima appears to depend principally upon some more simple geometrical truth, this latter has been separately premised; in order that it may be distinctly seen upon what elements each main proposition is founded. From a wish to accommodate this work, as far as it could well be done, to those who have studied only the first four books of Euclid, the doctrine of proportion has been, as much as possible, avoided: although the use of it might have shortened some of the demonstrations.

The propositions of the first and second Sections, of this first Part, form a distinct and important subject: they lead to results which have, most of them, been long known, but which are, perhaps, no where to be found collected, arranged, and strictly demonstrated. The maxima of the first Section are, each of them, connected with a minimum: that is, the same species of figure which renders the surface greatest when the perimeter is given, renders the perimeter least when the surface is given. This remarkable property is shewn, in a general theorem, necessarily to obtain. In the questions of the second Section, on the contrary, the area is a maximum when the perimeter is a maximum; and it is a minimum when the perimeter is a minimum. In one description of them, whilst the perimeter remains the same in length, the area also remains the same, whatever be the number of sides of the figure.

The third Section consists of miscellaneous propositions; classed, however, according to a division, which refers them to lines, angles, and surfaces.

It could not escape observation, if the mention of it were suppressed here, that it is part of the plan of this work to invite a comparison between Geometry and Algebra, and to illustrate the advantages peculiar to each. The relative advantages of these two great branches of science, in the investigation of mathematical truths, are now, indeed, well understood. But it may not be improper to

offer some remarks on the great difference which there is between them in producing those collateral effects, which have been ascribed to the mathematics, considered as a discipline of the mind.

In the very entrance upon our discussion of this topic, in order to avoid all misconstruction, it may not be wholly needless to state expressly, that what follows is intended to refer solely to the case of academical students, who apply themselves to the mathematics, not so much on account of the intrinsic value of that science itself, as for the sake of those indirect advantages, which are supposed to flow from the cultivation of it; such as the habits of close attention, of weighing the validity of proofs, of searching into the connexion of related truths, and of methodizing the materials of thought. With this particular view, then, let it be remembered, our enquiry is to be conducted: and it will turn principally on the comparative merits of the analytic and the synthetic modes of reasoning, so considered. Both these modes of reasoning may, indeed, be used in every department of the mathematics. But throughout the whole province of arithmetic, numerical as well as algebraical, elementary as well as infinitesimal, both in the investigation of theorems and in the solution of problems, the analytic method is, almost exclusively, employed: whilst the truths of Geometry are, for the most part, demonstrated synthetically; and the student in acquiring them becomes habituated to the use of that method of teaching. Now, there

exists, in the first place, this manifest distinction between a synthetic proof in Geometry, and an analytic process in Algebra, that in order to comprehend the former, the whole chain of reasoning must be kept in view, as it is continued from the beginning of the proposition to the end: whilst in pursuing the latter method, the attention is fixt only upon each single step, as each of them successively offers itself; and the conclusion is to be admitted independently of all but the last of them, whenever it is arrived at. Stronger and more unceasing attention, therefore, is required in the former case, than in the latter, and the judgment, as well as the memory, is called more urgently into action. There is, however, analysis, as well as synthesis, in Geometry. All those propositions, the truth of which Euclid has deduced *ex absurdo*, are, in reality, demonstrated analytically: and, in the same manner, a series of conclusions legitimately drawn from a certain supposition, may so terminate as to shew that supposition to be true. But, in both these cases, it is evident that the connexion of the several steps with the original hypothesis must be closely attended to, in order that the force of the proof may be clearly seen. Arithmetical speculations, on the contrary, most commonly hinge upon the solution of an equation, or upon the finding of a fluent: and whatever obstacles the authour of such propositions may have had to encounter, there is seldom any serious difficulty in following him along the path which he has traced out.

There is, besides, a more absolute precision in all the forms and in the language of geometrical disquisition. Pure Geometry is always precise and logical; it carries on its demonstrations by the exact comparison of ideas, adhering to the constant use of terms, the meanings of which are always verified by a reference to accurate definitions. Its reasonings proceed by means of syllogisms, in which, for the sake of brevity, the minor proposition is suppressed. But even in the proofs of those theorems of Algebra, in which little depends upon the employment of its peculiar symbols, the reasoning is seldom close and exact.

The absurdities which have been published with a view of explaining the rule for Algebraic multiplication; the common method of shewing that the numerator and denominator of a fraction in its lowest terms are measures of the numerator and denominator, respectively, of every other equal fraction; the imperfect state in which the proof of the rule for finding the greatest common measure of two complex algebraic quantities, has been left by most elementary writers, as if they could only be accurate as far as Euclid is accurate, from whom they have copied, but who did not contemplate the nature of quantities expressed algebraically; the defects in the demonstration of the binomial theorem; and many more examples might be adduced in support of the assertion made above. Nay, it is well known, that some propositions of the greatest importance in Algebra have never yet received a

satisfactory proof: and although mere metaphysical objections ought* not to stop the progress of any science, it is time that these faults were remedied: the most eminent writers in this department, however, always appear to be in haste to quit the province of severe reasoning, and to exhibit their skill in the management of symbols. Thus it appears, that even when the same method is used in both, Geometry affords a better exercise, than Algebra, for the mental powers. That it exercises, without oppressing and fatiguing them, will scarcely be denied by any man, of even middling abilities, from his own experience. Different individuals may, indeed, find it more or less difficult to retain, and to recollect, the proofs of a long series of geometrical propositions; but fully to comprehend these proofs, at the time when they are considered, to

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* "Quoique les vérités mathématiques soient toutes d'une certitude parfaite, elles n'ont pas toutes le même degré d'évidence: ce sont sur-tout les notions premières qu'il est difficile de porter au point de clarté desirable; mais on seroit arrêté dans la carrière, dès les premiers pas, si parce que certains principes fondamentaux restent enveloppés de quelque obscurité, on refusoit d'aller plus avant. Ainsi les anciens, parce qu'ils n'avoient pu parvenir à éclaircir entièrement la théorie des parallèles, n'ont point été pour cela retardés dans leurs recherches.

Ainsi, quoique la notion de l'infini présentât aux modernes des difficultés, ils n'ont pas laissé de donner à l'analyse infinitésimale tout le développement qu'on pouvoit attendre de leur sagacité. Ainsi, enfin, l'obscurité dans laquelle est restée la notion des quantités négatives n'a nullement entravé la marche des algébristes." *Carnot*, *Geom. de posit. p.* 481.

perceive the concatenation which binds the parts of the series together, and thus indirectly to gain all the essence of a system of logic, without the tediousness of its technical terms and rules, requires nothing more than common sense and sedulous application. What has been hitherto said, upon this subject of comparison, relates chiefly to the progress of the student in making himself master of the discoveries of other men. But it is not only in reading and digesting what has been written upon the mathematics, that his mind is disciplined: another most important employment of his faculties consists in the application of knowledge so acquired to cases which are new to him.

Now the questions proposed to a learner to be answered algebraically, as a trial of his skill and talents, are usually of such a kind as not to demand any extraordinary exertion of his reason. He is not called upon to attempt the investigation of new and recondite theorems. The difficulty presented to him, is seldom more than the mere translation of the conditions of the question, into a language, the peculiarity of which is, that it is so concise as to exhibit several propositions in a small compass. This having once been effected, and it is seldom an arduous task to perform, the attention is then withdrawn from the things signified, and confined to the signs: and from performing the mere operations of Algebra, it will scarcely be contended that any improvement of the reasoning faculties is to be derived. But the exercise of the understanding

is of a very different kind, when it is occupied in the solution of a geometrical problem. Whether it proceeds, strictly speaking, analytically, or whether it makes a particular construction, in the way of trial and conjecture, and then pursues the consequences of it, until they either end in the attainment of that which was proposed, or else indicate that some other method must be had recourse to, its faculties of judging, recollecting and inventing are continually exercised.

Algebra is, doubtless, the more powerful and convenient instrument for use. "Idem omnino mihi," says Euler, "cum Newtoni Principia et Hermanni Phoronomia perlustare cœpissem, usu venit, ut quamvis plurium problematum solutiones satis percepisse mihi viderer, tamen parum tantum discrepantia problemata resolvere non potuerim." But the same causes, which give analytics their superiority in that respect, prevent them from being so valuable, considered as a mental discipline. The great praise, it may be further remarked, which has been bestowed upon the Mathematics as conducing to strengthen the mind, has proceeded from men, who lived when Geometry constituted the principal part of them: and those who have lately denied them this merit, seem to have been biassed in their estimate by a partiality for extended analytics.

If the view which has here been taken of this subject be just, it should seem to be no disservice to our established system of education, to afford

scope for the efforts of our junior students in an easy extension of those rudiments of knowledge which they learn from Euclid. It is impossible for them to enter upon a more fertile field than that of Geometry, which really seems to admit of the exercise of as much genius and invention as poetry itself: and after having thus strengthened their faculties, and accustomed themselves to the comparison of clear ideas, they will proceed with better success to the remaining part of their academical course. A permanent taste for the Mathematics will thus be formed, and a study, which is now too frequently thrown aside as soon as it has answered a temporary purpose, will become a valuable resource to amuse and adorn their future leisure.

ON

# MAXIMA AND MINIMA.

## PART I.

ON THE APPLICATION OF EUCLID'S ELEMENTS TO QUESTIONS CONCERNING MAXIMA AND MINIMA.

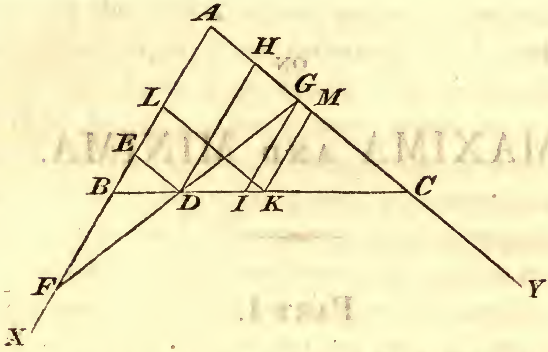
### SECTION I.

1. DEF. A VARIABLE magnitude is said to be a "maximum" when it is the greatest of its kind, or the greatest under certain conditions: and it is called a "minimum" when it is the least of its kind, or the least under certain conditions.

PROP. I.

2. *Problem.* Through a given point, situate between two given straight lines, which are not parallel, to draw a straight line terminated by the two given lines, and bisected in the given point.

Let $AX$ and $AY$ be two given straight lines which are not parallel, and $D$ a given point between them; it is required to draw a straight line through $D$, terminated by $AX$ and $AY$, and bisected in $D$.




Through $D$ draw (*E. 31. 1.) $DE$ parallel to $AY$, and $DH$ parallel to $AX$; from $HY$ cut off (E. 3. 1.) $HG$ equal to $DE$ or $AH$; join $G$, $D$, and produce $GD$ to meet $AX$ in $F$. The straight line $GF$ is bisected in $D$.

For the two triangles $FDE$, $DGH$ have the side $DE$ equal to the side $GH$, by the construction; and the angles adjacent to these sides are equal, each to each, because (E. 29. 1.) $DE$ is parallel to $AG$; wherefore (E. 26. 1.) $FD$ is equal to $DG$; and $FG$ is bisected in $D$.

## Prop. II.

3. *Theorem.* Of all triangles having the same vertical angle, and the bases of which pass through the same given point, the least is that which has its base bisected by the given point.

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* The letter E refers to the Elements of Euclid, the former of the two numbers to the Proposition, and the latter to the Book.

Let $BAC$ (Fig. Art. 2.) be the common vertical angle, and $D$ the point through which the bases of all the triangles pass. Draw (Art. 2.) the base $FG$ such that it shall be bisected in $D$, and let $BDC$ be the base of any other of the triangles; the triangle $AFG$ is less than the triangle $ABC$.

Through $G$, the extremity of $FG$ which is not further than the other extremity, from $A$, draw (E. 31. 1.) $GI$ parallel to $AB$ and meeting $BC$ in $I$; then, because $FD$ is equal to $DG$, and the angle $BDF$ to the angle $GDI$ (E. 15. 1.) and the angle $BFG$ to the angle $FGI$ (E. 29. 1.), the triangle $GDI$ is equal (E. 26. 1.) to the triangle $BDF$; but the triangle $GDI$ is less than the triangle $GDC$; therefore the triangle $BDF$ is less than the triangle $GDC$; if, therefore, the trapezium $ABDG$ be added to both, the triangle $AFG$ is less than the triangle $ABC$.

## Prop. III.

4. *Theorem.* The greatest parallelogram which can be inscribed in a given triangle, so as to have the vertical angle of the triangle for one of its angles, is that which is formed by drawing two straight lines from the bisection of the base, each parallel to a side of the triangle.

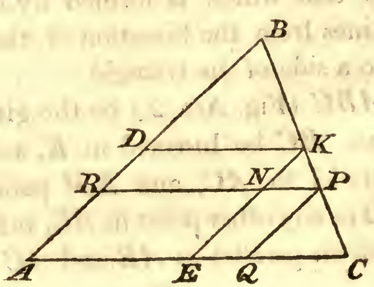
Let $ABC$ (Fig. Art. 2.) be the given triangle; let its base $BC$ be bisected in $K$, and let $KL$ be drawn parallel to $AC$, and $KM$ parallel to $AB$. Also let $D$ be any other point in $BC$, and let $DH$ and $DE$ be drawn parallel to $AB$ and $AC$ respectively;

the parallelogram *AK* is greater than the parallelogram *AD*.

Through the point *D* draw (Art. 2.) the straight line *FDG* so that it may be bisected in *D*; the triangles *BKL, KCM* may be shewn to be equal in the same manner as the triangles *BDF, GDI* were shewn to be equal in Art. 3.; therefore *LK* is equal to *MC*; and *LK* is also (E. 34. 1.) equal to *AM*; therefore *AM* is equal to *MC*; and (E. 41. 1.) the parallelogram *AK* is the double of the triangle *MKC*; it is, therefore, equal to the two triangles *MKC, LBK*, and is the half of the whole triangle *ABC*. In the same manner the parallelogram *AD* may be shewn to be half of the triangle *AFG*: but (Art. 3.) the triangle *ABC* is greater than the triangle *AFG*; therefore, also, the parallelogram *AK* is greater than the parallelogram *AD*.

## Prop. IV.

5. *Theorem.* Of all equiangular parallelograms of equal perimeters, that which is equilateral is the greatest.



Let $ADKE$ be any equilateral parallelogram, and $AP$ any other parallelogram equiangular with it, and of equal perimeter; $AK$ is greater than $AP$*.

Join $K$, $P$, and produce $KP$ both ways to meet $AD$ and $AE$, produced, in $B$ and $C$. And since $ND$ is a parallelogram, $RN$ is equal to $DK$, and $RD$ to $KN$ (E. 34. 1.); but $AD$ together with $DK$ is equal to $AR$ together with $RP$, each being the half (E. 34. 1.) of equal perimeters; from each of these equals take $AR$ together with $RN$ or $DK$, and there will remain $RD$ equal to $NP$; therefore, also $NK$ is equal to $NP$; therefore the angle $NKP$ is equal to the angle $NPK$ (E. 5. 1.); but the angle $NKP$ is equal to the angle $ABC$, and angle $NPK$ to the angle $ACB$ (E. 29. 1), and also to the angle $DKB$; wherefore the four angles $DBK$, $DKB$, $EKC$, $ECK$ are equal to each other; and the side $DK$ of the triangle $BDK$ is, by the hypothesis, equal to the side $KE$ of the triangle $KEC$; therefore (E. 26. 1.) $BK$ is equal to $KC$; and (Art. 4.) $AK$ is greater than $AP$.

6. Cor. 1. The square is the greatest of all

* If it be required to make an equilateral parallelogram equiangular with a given parallelogram $AP$, and having also an equal perimeter; produce $AQ$ to $C$, and make $QC$ equal to $QP$. Join $C$, $P$; and let $CP$ produced meet $AR$ produced in $B$; bisect (E. 10. 1.) $CB$ in $K$; and through $K$ draw (E. 31. 1.) $KD$ and $KE$ parallel to $AQ$ and $AR$ respectively; then it is manifest, from the demonstration of Art. 5., that $AK$ is equilateral and that its perimeter is equal to that of the equiangular parallelogram $AP$.

rectangles of equal perimeters; which may also be deduced from E. 5. 2.

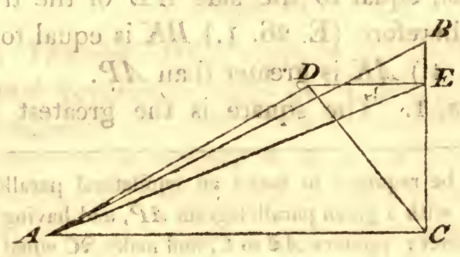
7. COR. 2. The space which can be enclosed by a straight line of given length, and an indefinite straight line, the given finite line being divided into two segments, which are to be inclined to each other at a given angle, is greatest when the segments are equal.

For (Art. 5.) the double of that space will be a maximum when the segments are equal.

PROP. V.

8. *Theorem.* Of all triangles having two sides of the one equal to two sides of the other, each to each, that which has the two sides perpendicular to each other is the greatest.

Let the triangle *ACB* have the two sides *AC*, *CB* at right angles to each other, and let *ACD*




be any other triangle standing upon *AC*, and having the other side *CD* equal to *CB*. The triangle *ACB* is greater than the triangle *ACD*.

Draw *DE* (E. 12. 1.) perpendicular to *BC* and

join $A$, $E$. Then, because the angle $DEC$ is a right angle, the angle $EDC$ is less than a right angle (E. 17. 1.); and, therefore, (E. 19. 1.) $DC$ is greater than $EC$; but $DC$ is equal to $BC$; therefore $BC$ is greater than $EC$; again, because each of the angles $DEC$, $ECA$ is a right angle, $DE$ (E. 28. 1.) is parallel to $AC$, and the triangle $ADC$ is equal (E. 37. 1.) to the triangle $AEC$. But the triangle $ABC$ is greater than the triangle $AEC$, because $BC$ has been shewn to be greater than $EC$; therefore the triangle $ABC$ is greater than the triangle $ADC$.

9. Cor. 1. A square is greater than any other given parallelogram of equal perimeter: And the perimeter of a square is less than that of any other equal parallelogram.

If the given parallelogram be not equilateral, find (Note, Art. 5.) an equilateral parallelogram equiangular with it, and of equal perimeter.

This is (Art. 5.) greater than the given parallelogram; and if the given parallelogram be rectangular, it will be a square: if not, it will be a rhombus: but (Art. 8.) the half of a square is greater than the half of a rhombus of equal perimeter; wherefore the whole square is greater than the whole rhombus; much more then is the square greater than the given parallelogram of equal perimeter.

Conversely, let $A$ be a square, and $B$ any other equal parallelogram; the perimeter of $A$ is less than that of $B$. For if it be not less, it is either equal to it or greater; but it cannot be equal; for

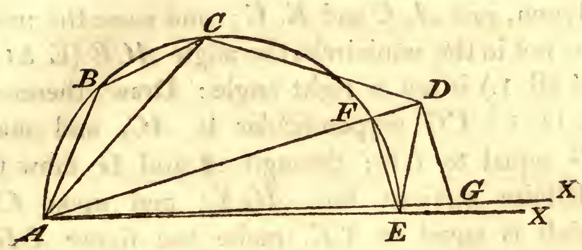
then, as hath been shewn, $A$ would be greater than $B$; which is contrary to the supposition: Neither can the perimeter of $A$ be greater than that of $B$; for then $A$ would manifestly be greater* than a square of equal perimeter with $B$, the which square, as hath been proved, is itself greater than $B$; much more, then, would $A$ be greater than $B$; which is contrary to the supposition. Wherefore, the perimeter of the square $A$ is less than that of any other equal parallelogram $B$, since it can neither be equal to it, nor greater than it.

10. Cor. 2. The space which can be enclosed by a straight line of a given length, and an indefinite straight line, the given finite line being divided into two segments, is greatest when the segments are equal, and perpendicular to each other.

11. Cor. 3. If space be to be enclosed by any number of given finite straight lines together with an indefinite straight line, and if the semicircle described upon the assumed portion of the indefinite line as a diameter do not pass through all the angular points of the figure, a greater space may be enclosed under the same conditions.

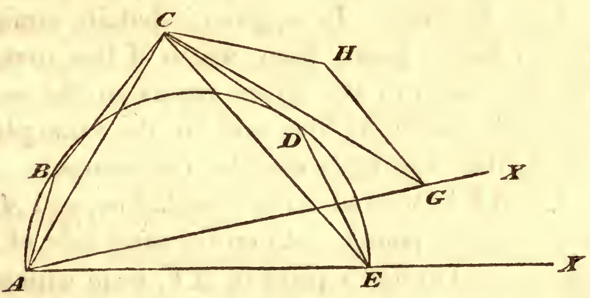
Let $AB$, $BC$, $CD$, $DE$ be the given finite straight lines, placed so as to enclose a space with the indefinite straight line $AX$; and let the semicircle $ABCE$, described upon $AE$ as a diameter,

* It is self-evident, that of two squares, that which has the greater side is the greater, and conversely: That this is true of all regular polygons of the same number of sides, is shewn in Art. 23.




pass through the angular points $B$ and $C$, but not through $D$; join $A$, $D$ and $E$, $F$; draw (E. 11. 1.) $DG$ perpendicular to $AD$, and make $DG$ equal to $DE$; join $A$, $G$; the angle $EFD$ is a right angle (E. 31. 3.); therefore (E. 17. 1.) the angle $ADE$ is not a right angle; therefore (Art. 8.) the triangle $ADG$ is greater than the triangle $ADE$; if, therefore, the figure $ABCD$ be added to both, $ABCDG$ is greater than $ABCDE$.

But, secondly, if the semicircle $ABDE$, described upon $AE$ as a diameter, pass through the




angular points $B$ and $D$, but not through $C$, which point is not adjacent either to the first or the last of the given finite straight lines, which bound the

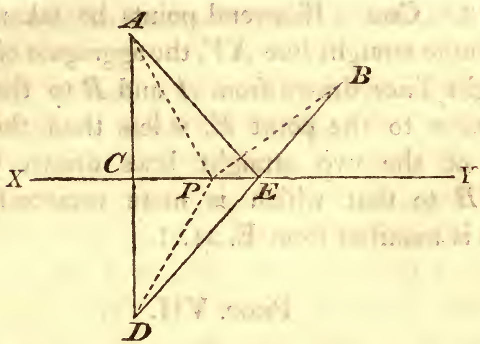
polygon, join *A, C* and *E, C*; and since the point *C* is not in the semicircle, the angle *ACE* (E. 31. 3. and 16. 1.) is not a right angle: Draw, therefore, (E. 11. 1.) *CG* perpendicular to *AC*, and make *CG* equal to *CE*; through *A* and *G* draw the indefinite straight line *AGX*; and upon *CG*, which is equal to *CE*, make the figure *CHG* (E. 18. 6.) similar, and therefore (E. 20. 6.) equal to *CDE*, i. e. to that part of the polygon which stands on *CE*: Then, since (Art. 8.) the triangle *ACG* is greater than the triangle *ACE*, if to the former be added the figures *ABC* and *CHG*, and to the latter the figures *ABC* and *CDE*, which has been shewn to be equal to *CHG*, it is manifest that the whole figure *ABCHG* is greater than the whole figure *ABCDE*, which is enclosed under the same conditions.

## Prop. VI.

12. *Problem.* In a given indefinite straight line, to find a point, from which if two straight lines be drawn to two given points on the same side of the indefinite line, and in the same plane with it, their aggregate shall be a minimum.

Let *XY* be the indefinite straight line, and *A, B* the two given points, both on the same side of it; it is required to find a point in *XY*, from which, if two straight lines be drawn to *A* and *B*, their aggregate shall be a minimum.

From *A* draw the straight line *ACD* perpendicular to *XY*, and make *CD* equal to *AC*; join




$D$, $B$, and let $DB$ meet $XY$ in $E$; also join $E$, $A$; $AE$, together with $EB$, is a minimum.

For, let $P$ be any other point in $XY$, and join $P$ with $A$, $B$, $D$. Because $AC$ is equal to $CD$, and $CE$ common to the two triangles $ACE$, $DCE$, and that the angle $ACE$ is equal to the angle $ECD$ (E. Def. 10. 1.) $AE$ is equal (E. 4. 1.) to $DE$; and in the same manner $AP$ may be shewn to be equal to $DP$; therefore $DB$ is equal to $AE$ together with $EB$, and $DP$ together with $BP$ is equal to $AP$ together with $PB$; but $DP$ together with $PB$ is greater (E. 20. 1.) than $DB$; therefore, also, $AP$ together with $PB$ is greater than $AE$ together with $EB$*.

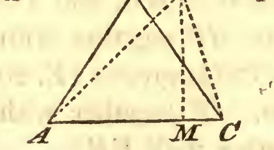
* If $XY$ be the circumference of a great circle in a sphere, and if $A$, $B$, be two given points, in the sphere's surface, both of them on the same side of $XY$, the same kind of construction may be made as in the above proposition, and a point in $XY$ may be thereby determined, such that the aggregate of the arches of two great circles, drawn to it from $A$ and $B$, shall be

13. Cor. If several points be taken in the indefinite straight line $XY$, the aggregate of the two straight lines drawn from $A$ and $B$ to that which is nearer to the point $E$, is less than the aggregate of the two straight lines drawn from $A$ and $B$ to that which is more remote from $E$. This is manifest from E. 21. 1.

## Prop. VII.

14. *Theorem.* The perimeter of an isosceles triangle is less than that of any other equal triangle, standing upon the same base.

Let $ABC$ be an isosceles triangle, and $AQC$ any other equal triangle standing upon the same

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be a minimum. See a Treatise on *Spherics*, by the Authour of this Treatise (Art. 70. 92. 66. 97. 74.).

It may, perhaps, in many instances, furnish amusement to the reader, whilst he is engaged with the problems in this book, to endeavour to solve the analogous problems, on the surface of a sphere, by an application of the principles of Spherical Geometry.

base $AC$; the perimeter of $ABC$ is less than that of $AQC$.

Join $B$, $Q$ and produce $BQ$ both ways to $X$ and $Y$; $BQ$ is parallel (E. 39. 1.) to $AC$, and, therefore, the angle $ABX$ is equal to the alternate angle $BAC$, and the angle $CBY$ to the angle $BCA$ (E. 29. 1.); but the angle $BAC$ is equal (E. 5. 1.) to the angle $BCA$; therefore the angle $ABX$ is equal to the angle $CBY$; it follows, therefore, from the demonstration of Art. 12., that $AB$ together with $BC$ is less than $AQ$ together with $QC$; and, because the base $AC$ is common to the two triangles, the perimeter of the triangle $ABC$, is less than that of the triangle $AQC$.

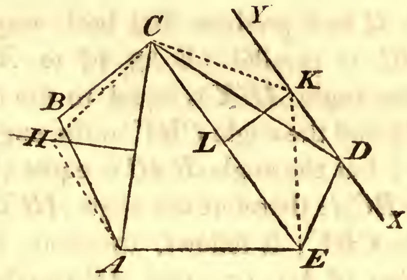
Otherwise:

Let $AB$ be produced to $D$ so that $BD$ may be equal to $AB$, or $BC$, and let $D$, $Q$ be joined; the angle $DBQ$ (E. 29. 1.) is equal to $BAC$, and $QBC$ to $BCA$; but $BAC$ is equal (E. 5. 1.) to $BCA$; wherefore $DBQ$ is equal to $QBC$; and $BD$ is equal to $BC$, and, therefore, (E. 4. 1.) $DQ$ is equal to $QC$; but $AQ$ together with $QD$, is greater (E. 20. 1.) than $AD$; i. e. than $AB$, $BC$; therefore $AQ$, together with $QC$, is greater than $AB$, together with $BC$; and the base $AC$ is common to the two triangles; therefore the perimeter of the triangle $ABC$ is less than the perimeter of the triangle $AQC$.

15. Cor. 1. If any polygon be not equilateral, another equal polygon may be found, of the same number of sides, which has a less perimeter.

Let the two sides $CD$, $DE$ of the polygon

*ABCDE* be unequal; join *C, E* and (E. 10. 1.)




bisect *CE* in *L*; through *D* draw (E. 31. 1.) *XY* parallel to *CE*, and through *L* (E. 31. 1.) *LK* perpendicular to *CE*; and join *C, K* and *E, K*. By the construction, and E. 4. 1., the two triangles *CLK*, *ELK* are equal; so that *CK* is equal to *KE*, and therefore, (Art. 14.) *CK*, together with *KE*, is less than *CD*, together with *DE*; add to both the rest of the perimeter *EABC*, and the whole perimeter *EABCK* is less than the whole *EABCD*.

16. Cor. 2. An isosceles triangle is greater than any scalene triangle of equal perimeter, and standing upon the same base.

If not, it is either equal to, or less than the scalene triangle; but it cannot be equal to the scalene triangle; because it would then (Art. 14.) have a less perimeter, which is contrary to the hypothesis; neither can it be less; because then, also, its perimeter must be still less than before; for if two unequal isosceles triangles stand upon the same base, and on the same side of it, the less must lie wholly within the greater; and, therefore,

(E. 21. 1.) have a less perimeter; otherwise, an angle at the base, belonging to one of the triangles, would, at the same time, be greater and less than an angle at the base belonging to the other. Wherefore the isosceles triangle is greater than the scalene triangle, since it can neither be equal to it, nor less than it.

17. Cor. 3. Hence, if any polygon be not equilateral, a greater polygon may be found of the same number of sides, and of equal perimeter.

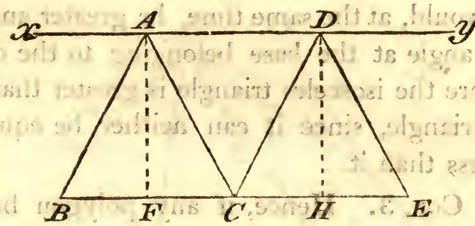
Let the sides *AB*, *BC* of the polygon *ABCDE* (Fig. Art. 15.) be unequal; join *A*, *C*; from the centre *A*, at a distance equal to the semi-aggregate of the two sides *AB*, *BC*, describe a circle, and let it cut another equal circle, described from the centre *C*, at an equal distance, in *H*. Join *A*, *H* and *C*, *H*; the isosceles triangle *AHC* is, therefore, of the same perimeter as *ABC*, and (Art. 16.) is greater than *ABC*; add to each the polygon *ACDE*, and *AHCDE* is greater than *ABCDE*, a polygon of the same number of sides, and of equal perimeter.

## Prop. VIII.

18. *Theorem.* If the bases of two equal isosceles triangles be equal, the side, also, of the one is equal to the side of the other.

Let *ABC*, *DCE* be two equal isosceles triangles, having the base *BC* of the former equal to

the base $CE$ of the latter; the side $AB$ is equal to




the side $DC$.

Let the bases be placed contiguously at the extremities and in the same straight line; join $A$, $D$ and from $A$ and $D$ draw (E. 12. 1.) $AF$ and $DH$ perpendicular to $BE$.

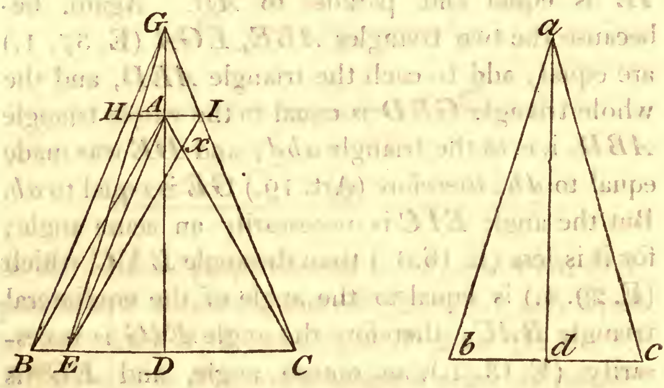
Then (E. 40. 1.) $AD$ is parallel to $BE$, and (E. 28, and 29. 1.) $AH$ is a parallelogram; also, because $AF$ is common to the two right-angled triangles $AFC$, $AFB$, which have the angles at $B$ and $C$ equal, $BF$ (E. 26. 1.) is equal to $FC$; in the same manner, $CH$ is equal to $HE$; therefore $FC$ is equal to $CH$, each being the half of two equal lines; and $FH$ is equal to $BC$; but (E. 34. 1.) $FH$ is equal to $AD$; therefore, also, $BC$ is equal to $AD$, and (E. 33. 1.) $AB$ is equal to $DC$.

19 Cor. In the same manner it may be shewn that if the bases of two equal right-angled triangles be equal, the remaining sides of the one are equal to the remaining sides of the other, each to each.

## Prop. IX.

20. If the base of an isosceles triangle be less than the base of an equal equilateral triangle, its side shall be greater than the side of the equilateral triangle.

Let $a\,b\,c$ be an isosceles triangle, and $ABC$ an




equilateral triangle equal to it; and let $b\,c$ be less than $BC$; then is $a\,b$ greater than $AB$.

From $a$, and $A$, draw $a\,d$ perpendicular to $b\,c$, and $AD$ perpendicular to $BC$. It may be shewn, as in the preceding proposition, that $b\,c$ and $BC$ are bisected in the points $d$ and $D$; therefore, the triangle $ABD$ is equal to the triangle $a\,b\,d$, each (E. 38. 1.) being the half of the two equal triangles. From $DB$ cut off (E. 3. 1.) $DE$ equal to $d\,b$; join $A$, $E$; through $B$ draw (E. 31. 1.) $BG$ parallel to $EA$, and let it meet $DA$ produced in $G$; also, through $A$ draw $HAI$ parallel to $BC$, and join $G$, $E$ and $G$, $C$. Then, (E. 4. 1.) the angle

*GBC* is equal to the angle *GCB*; and, therefore, because the two triangles *GAH*, *GAI* have the angles at the bases *HA* and *IA* equal (E. 29. 1.) and have the side *GA* common, *HA* is equal (E. 26. 1.) to *AI*; but *HA* is also equal to *BE* (E. 34. 1.), because *HE* is a parallelogram; therefore *AI* is equal to *BE*, and (E. 33. 1.) *IE* is equal and parallel to *AB*. Again, be-because the two triangles *ABE*, *EGA* (E. 37. 1.) are equal, add to each the triangle *AED*, and the whole triangle *GED* is equal to the whole triangle *ABD*, i. e. to the triangle $abd$; and *DE* was made equal to $db$; therefore (Art. 19.) *GE* is equal to $ab$. But the angle *EIC* is necessarily an acute angle; for it is less (E. 16. 1.) than the angle *EXC*, which (E. 29. 1.) is equal to the angle of the equilateral triangle *BAC*; therefore the angle *EIG* is necessarily (E. 13. 1.) an obtuse angle, and *EG* is greater than (E. 17. and 19. 1.) *EI*. But *EG* was shewn to be equal to $ab$, and *EI* to *AB*. Therefore $ab$ is greater than *AB*.

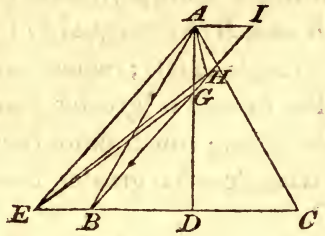
PROP. X.

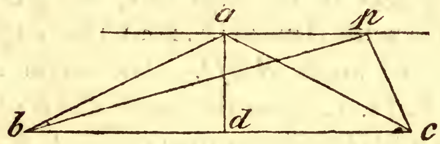
21. *Theorem.* The perimeter of an equilateral triangle is less than the perimeter of any other equal triangle.

Let *ABC* be an equilateral triangle, and $pbc$ any other equal triangle; the perimeter of the triangle *ABC* is less than that of the triangle $pbc$.

If the triangle $pbc$ be not isosceles, find (Art.

15.) an equal isosceles triangle *a b c*, which will have a less perimeter; draw (E. 12. 1.) *AD* perpendicular




to *BC*, and *a d* perpendicular to *b c*; and from *DB*, produced if necessary, cut off (E. 3. 1.) *DE* equal to *d b*; join *A*, *E*; through *B* draw (E. 31. 1.) *BG* parallel to *EA*, and through *A* draw *AI* parallel to *BC*, meeting *BG* produced in *I*; and join *E*, *G*. It may be shewn, as in Art. 20., that the triangle *EGD* is equal to the triangle *ABD*, and that, therefore, (Art. 19.) *EG* is equal to *a b*. And, first, if *ED* be not less than *DA*, the angle *DAE* is not less (E. 5. and 18. 1.) than the angle *AED*, and, therefore, the angle *AGI* is not less than the angle *AIG* (E. 29. and 34. 1.); therefore (E. 6. and 19. 1.) *AI* or *EB* is not less than *AG*. But *EG* and *GA* are together (E. 20. 1.) greater than *EA*, and *EA* is greater (E. 19. 1.) than *AB*, because the angle *ABE* is necessarily obtuse; much more then are *GE* and *EB* together greater than

*AB*; add *BD* to both; therefore *GE* and *ED* are together greater than *AB* and *BD*. But *GE* and *ED* are equal to the semi-perimeter of the triangle *a b c*, and *AB* and *BD* are equal to the semi-perimeter of the triangle *ABC*; wherefore the whole perimeter of the former is greater than the whole perimeter of the latter; much more then is the perimeter of the triangle *p b c* greater than that of the triangle *ABC*.

But if *ED* be less than *DA*, and greater than *DB*, at the point *A* make (E. 23. 1.) the angle *EAH* equal to the angle *AED*. The angle *AED* is greater (E. 18. 1.) than the angle *DAE*; therefore the angle *EAH* is greater than the angle *EAD*, and *AH* and *AE* lie on contrary sides of *AD*; again, the angle *AED* is less than the angle *EAC*, for *AC*, which is equal to *BC*, is, by the hypothesis, less than *CE*; therefore the angle *EAH* is less than the angle *EAC*; and *AH* falls between *AD* and *AC*; therefore *AH* meets *BI* between *AD* and *AC*; join *E*, *H*. The angle *AHI* is equal (E. 29. 1.) to the angle *EAH*; i. e. to the angle *AEB*, by the construction. And the angle *AEB* is equal (E. 34. 1.) to the angle *AIH*; therefore (E. 6. 1.) *AI* is equal to *AH*. But the exterior angle *BGD* is greater (E. 16. 1.) than the angle *BAG*; and the angle *BGD* is equal (E. 15. 1.) to the angle *AGH*, and the angle *BAG* is equal to the angle *GAC*, because the perpendicular *AD* bisects the triangle *BAC*; but the angle *GAC* has been shewn to be greater than the angle *GAH*; much more, then, is the

angle $AGH$ greater than the angle $GAH$; therefore (E. 19. 1.) $AH$, or $AI$, or (E. 34. 1.) $EB$ is greater than $GH$; but the two $EG$, $GH$ are together (E. 20. 1.) greater than $EH$; much more, then, are $GE$, $EB$ together greater than $EH$. And since $AH$ is equal to $EB$, and $HB$ common to the two triangles $HAB$, $HEB$, and that the angle $AHB$ is equal to the angle $EBH$, the side $EH$ is equal (E. 4. 1.) to the side $AB$; wherefore $GE$ and $EB$ are together greater than $AB$, and it may be shewn, as in the first case, that the perimeter of the triangle $ABC$ is less than the perimeter of the triangle $a\,b\,c$; much more, then, is it less than that of the triangle $p\,b\,c$.

Lastly, if the base of the isosceles triangle be the less of the two, its side is greater (Art. 20.) than the side of the equilateral triangle. And if on one of the sides of this isosceles triangle, another isosceles triangle be constructed, equal to it, as in Art. 15., its perimeter shall (Art. 14.) be less than that of the former isosceles triangle; and the perimeter of this latter triangle is greater than that of the equal equilateral triangle, by the preceding cases. Therefore, in this case, also, the perimeter of the equilateral triangle is less than that of the given triangle.

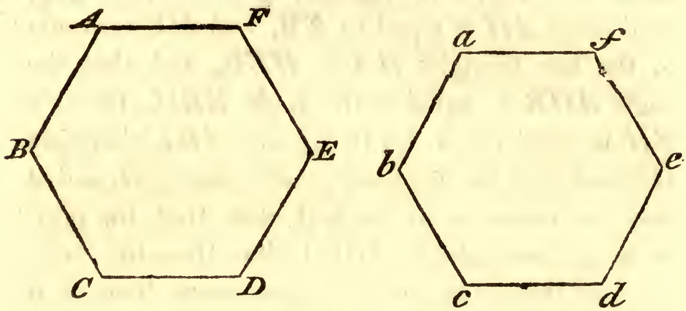
22. Def. A regular polygon is a plane rectilineal figure which is equilateral and equiangular.

### Prop. XI.

23. *Theorem.* Of all regular polygons, con-

tained by the same number of sides, that which has the greatest perimeter is the greatest; and that which is the greatest has the greatest perimeter.

Let *AD*, *a d* be two equilateral and equiangular



polygons, and let *AD* be greater than *a d*; the perimeter also of *AD* is greater than that of *a d*.

For (E. Cor. 1. 32. 1.) the angles of the one figure are equal to the angles of the other, each to each; and since, the figures are both of them equilateral, they are (E. Def. 1. 6.) similar to each other. Wherefore (E. 20. 6.) *AD* is to *a d* as the square on *CD* is to the square on *c d*; but *AD* is greater than *a d*; therefore (E. 16. and 14. 5.) the square on *CD* is greater than the square on *c d*, and *CD* is greater than *c d*; but the perimeter of *AD* is the same multiple of *CD* as the perimeter of *a d* is of *c d*; therefore (E. 15. 5.) the perimeter of *AD* is greater than that of *a d*.

The converse of the proposition is proved in the same manner.

## Prop. XII.

24. *Theorem.* If the perimeter of a regular polygon be less than the perimeter of any other equal rectilineal figure of the same number of sides, the regular polygon is greater than any other rectilineal figure of the same number of sides, and of equal perimeter: And, conversely, if a regular polygon be greater than any other polygon of the same number of sides and of equal perimeter, then is its perimeter less than that of any other equal polygon of the same number of sides.

Let $A$ be a regular polygon, and $B$ a rectilineal figure of the same number of sides and of equal perimeter; then if the perimeter of $A$ be less than that of any other equal rectilineal figure of the same number of sides, $A$ is greater than $B$.

For if it be not greater, it is either equal to it or less; but it cannot be equal; for then, by the supposition, its perimeter would be less than, and not equal to, that of $B$; neither can it be less, for then its perimeter must be less (Art. 23.) than if it equalled $B$, and, therefore, less than the perimeter of $B$; therefore $A$ is greater than $B$, since it can neither be equal to it, nor less than it.

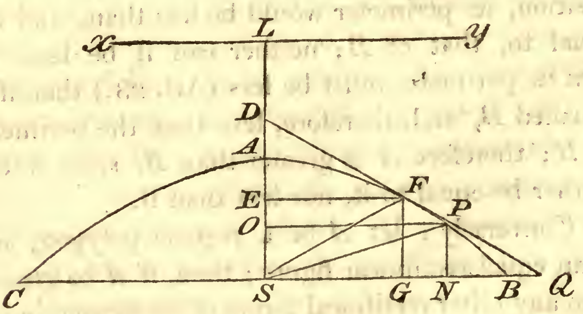
Conversely; let $A$ be a regular polygon, and $B$ an equal rectilinear figure; then, if $A$ be greater than any other rectilineal figure of the same number of sides and of equal perimeter, the perimeter of $A$ is less than that of $B$.

For if it be not less, it is either equal to it, or

greater; but it cannot be equal; for then, by the hypothesis, *A* would be greater than *B*; but it is also equal to *B*; which is absurd: neither can the perimeter of *A* be greater than that of *B*; for then, if *C* be another regular polygon, of the same number of sides, having its perimeter equal to that of *B*, and therefore less than that of *A*, it follows from Art. 23. that *A* is greater than *C*; and *C*, by the hypothesis, is greater than *B*; much more, then, is *A* greater than *B*; but it is also equal to *B*; which is absurd. Since, therefore, the perimeter of *A* can neither be equal to that of *B*, nor greater than it, the perimeter of *A* is less than that of *B*.

25. Cor. An equilateral triangle is greater than any other triangle of equal perimeter*.

* If any point be taken in the arch of a parabola, included between the vertex and focal ordinate, the distance of that point from the focus, together with its perpendicular distance from the focal ordinate, is equal to the distance between the focus




and the vertex. From this property Art. 25. may be thus deduced. Take the straight line *SB* equal to the semi-perimeter, of

For its perimeter was shewn (Art. 21.) to be less than the perimeter of any other equal triangle.

## Prop. XIII.

26. *Theorem.* The perimeter of a square is less than that of any other quadrilateral rectilineal figure which is equal to the square.

Let $ABCD$ be any quadrilateral rectilineal figure which is not a square; the perimeter of an equal square is less than that of $ABCD$.

Join $A$, $C$; and if $AB$ be not equal to $BC$, nor $AD$ to $CD$, bisect (E. 10. 1.) $AC$ in $E$; through

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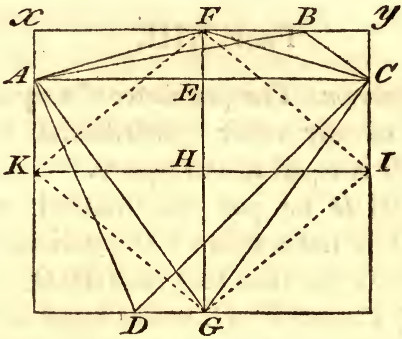
of any isosceles triangle; draw $SA$ perpendicular to $SB$, and make it equal to the half of $SB$; let the parabola $CAB$ be described, having its focus in $S$, and its vertex in $A$, and let $XY$ be its directrix. Let $SA$, produced, meet $XY$ in $L$: trisect $LS$ in the points $D$ and $E$; through $E$ draw $EF$ parallel to $SB$, and $FG$ perpendicular to $SB$. Join $S$, $F$: then, by the property of the curve, $SF$ is the side of an equilateral triangle of the given perimeter; and if any point $P$ be taken in $AFB$, and $S$, $P$ be joined, $SP$ is the side of an isosceles triangle of the same given perimeter. Draw $PN$ perpendicular to $SB$, and $PO$ parallel to $SB$. The rectangle $EG$ is equal to the equilateral triangle, and the rectangle $ON$ to the isosceles triangle. And since $AL$ is equal to $AS$, and $LD$ to $ES$, $DA$ is equal to $EA$, and, therefore, the straight line joining $D$, and $F$, will touch the curve in $F$; join $D$, $F$; and let $DF$ meet $SB$ produced in $Q$; because $DE$ is equal to $ES$, and $DF$ (E. 2. 6.) is equal to $FQ$; therefore (Art. 4.) the rectangle $EG$ is greater than the rectangle $ON$; i. e. the equilateral triangle is greater than any isosceles triangle of equal perimeter, and is, therefore, (Art. 16.) greater than any other triangle of equal perimeter.

*B* and *D* draw (E. 31. 1.) *BF* and *DG* parallel to *AC*, and through *E* draw (E. 11. 1.) *FEG* per-




pendicular to *AC*, and let it meet *BF* and *DG* in *F* and *G*; join *A*, *F*, and *C*, *F*, and *A*, *G*, and *C*, *G*; then (Art. 14, 15.) the perimeter of *AFCG* is less than that of *ABCD*. Again, if *AF* be equal to *AG*, the figure *AFCG* is equilateral; but if *AF* be not equal to *AG*, bisect *FG* in *H*; through *A* and *C* draw *AK*, and *CI*, parallel to *FG*, and through *H* draw *KHI* perpendicular to *FG*; join *F*, *K*, and *K*, *G*, and *G*, *I*, and *I*, *F*. The figure *FKGI* is (Art. 14, 15.) equal to *FAGC* and has a less perimeter; therefore, also, it is equal to *ABCD*, and has a less perimeter than *ABCD*; but *KG* is equal to *GI*, (E. 4. 1.) for *KH* is equal to *HI*, and *HG* is common to the two triangles *KHG*, *IHG*, which are right-angled at *H*; therefore the figure *FKGI* is equilateral, and is either a rhombus or a square; if it be a rhombus its perimeter is greater (Art. 9.) than that of an

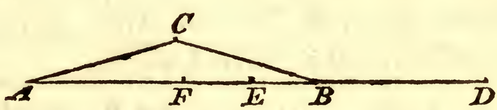
equal square; wherefore the perimeter of the square is less than that of the equal quadrilateral figure *ABCD*.

27. Cor. The square is greater than any other quadrilateral rectilineal figure of an equal perimeter (Art. 26. and 24.).

## SCHOLIUM.

28. What has been legitimately proved of the trapezium and the triangle (Art. 27. and 25.) namely, that the perimeter being given, the area is a maximum when the figure is equilateral, is usually inferred from Art. 17. to be true of all polygons whatever. It has been shewn, in the Introduction, that this mode of reasoning is defective. Before such a conclusion had been drawn, it ought, at least, to have been shewn, that, by repeating the process described in Art. 17., the sides of the polygon approximated to a ratio of equality. This may be readily proved to obtain in the case of the triangle, either by geometry, or algebraically.

Let *CAB* be the isosceles triangle resulting from




the first operation (Art. 17.) upon the given triangle. Produce *AB* to *D*, and make *BD* equal to *BC*. Bisect *AD* in *E*; then is *AE*, or *ED*, equal to the side of the next isosceles triangle to be

described, in a similar manner, upon $AC$, having its perimeter equal to that of $ACB$. From $AB$, produced if necessary, cut off $AF$ equal to $AC$; then, since $AE$ is equal to $ED$, and $AF$ to $BD$, $FE$ is equal to $EB$; and, therefore, $FB$ is the double of $FE$; but $FB$ is the difference between $AB$ and $AC$; and $FE$ is the difference between $AC$ and the side of the next isosceles triangle described on $AC$ as a base. Thus the difference between the base and the side of the resulting isosceles triangles is halved at each step; and, therefore, the three sides may be made to approach indefinitely near to a state of equality. This proof is equally applicable to the case of any three magnitudes whatever, whether any two of them be greater than the third or not; which may also be shewn algebraically.

For, let $a, b, c$ represent any series of three magnitudes; let a second series be found from it, by taking the semi-sum of $a$ and $b$ for the two first terms, and $c$ for the third; in the same manner let a third series be formed from the second, by taking the semi-sum of its two unequal terms, for the two first terms of this new series, and so on. The several series thus derived will be,

$$\text{I.}\quad \frac{a+b}{2},\ \frac{a+b}{2},\ c;\ \text{whence } c\sim\frac{a+b}{2}=\frac{2c\sim(a+b)}{2}.$$

$$\text{II.}\quad \frac{a+b+2c}{4},\ \frac{a+b+2c}{4},\ \frac{a+b}{2};$$

whence $\frac{a+b}{2} \sim \frac{a+b+2c}{4} = \frac{2c \sim (a+b)}{4}$.

III. $\frac{3a+3b+2c}{8}$, $\frac{3a+3b+2c}{8}$, $\frac{a+b+2c}{4}$;

whence, $\frac{3a+3b+2c}{8} \sim \frac{a+b+2c}{4} = \frac{2c \sim (a+b)}{8}$,

&c. &c. ..........................................

$n.\ \frac{\frac{2^n \mp 1}{3}.(a+b) + \frac{2^n \pm 2}{3}.c}{2^n}$, $\frac{\frac{2^n \pm 1}{3}.(a+b) + \frac{2^n \mp 2}{3}.c}{2^n}$,

$\frac{\frac{2^{n-1} \pm 1}{3}.(a+b) + \frac{2^{n-1} \mp 2}{3}.c}{2^{n-1}}$, and the difference between the two unequal terms is $\frac{2c \sim (a+b)}{2^n}$.

Let $n$ be considered as indefinitely great; then the limit of the terms of the $n^{\text{th}}$ series is evidently $\frac{a+b+c}{3}$.

Thus, by continuing the process, the terms of the resulting series may be made to approximate indefinitely to a state of equality. The same kind of reasoning may be extended to shew, that if any number of magnitudes be treated in the same manner, the aggregate of their differences will be halved at each step; it follows, therefore, that, being so treated, they will approximate indefinitely to a state of equality; and, since it has been proved that, if the sides of any polygon be so treated, the

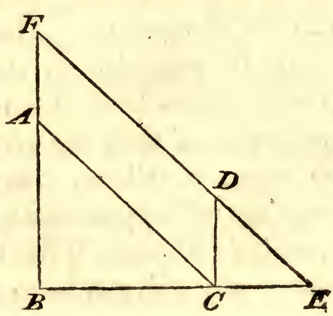
figure is at each step enlarged, whilst its perimeter remains the same, it may be thence inferred that the greatest polygon of a given perimeter, and a given number of sides, is equilateral.

The common mode of proving that this greatest polygon is also equiangular, as well as equilateral, is still more objectionable, upon the same grounds. The method of PAPPUS, which may be seen in the propositions immediately following, is very different; it is strict to the extent, to which it is here carried, and ingenious, although it be not concise.

## PROP. XIV.

29. *Theorem.* If the angles of the two right-angled triangles be equal, each to each, the square described upon the aggregate of the two hypotenuses is equal to the square described upon the aggregate of the bases, together with the square described upon the aggregate of the two remaining sides.

Let *ABC, DCE* be two triangles, having the



angles *ABC, DCE* right angles, the angle *BCA* equal to the angle *CED*, and the angle at *A* equal to the angle at *D*. The square described on the aggregate of *AC* and *DE* is equal to the square described on the aggregate of *BC* and *CE*, together with the square described on the aggregate of *BA* and *CD*.

Let the bases *BC, CE* be placed in the same straight line. It may be shewn, as in E. 4. 6., that *BA* and *ED* meet when produced, and that if they be produced to meet in *F*, the figure *FC* is a parallelogram; therefore (E. 34. 1.) *FD* is equal to *AC*, and *FA* to *DC*; but (E. 47. 1.) the square described on *FE*, is equal to the squares described on *FB, BC*; that is, to the square described on the aggregate of *AB, DC*, together with that described on *BC, CE*.

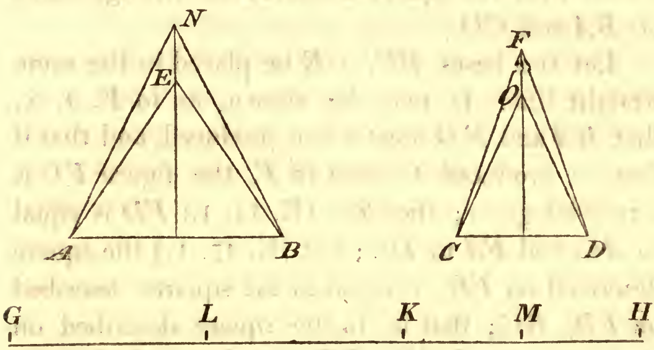
### Prop. XV.

30. *Problem.* Two isosceles triangles being given, which stand upon unequal bases, and are not similar to each other, to describe, upon the same bases, two other isosceles triangles similar to each other; and having their perimeters, taken together, equal to the perimeters of the two given triangles.

Let *AEB, CFD* be two dissimilar isosceles triangles, standing upon the unequal bases *AB, CD*, of which *AB* is the greater; it is required to describe upon *AB* and *CD* two isosceles triangles, which shall be similar to each other, and shall

have their perimeters, taken together, equal to the perimeters of the two given triangles.

Take the straight line *GH*, and make it equal to *AE*, *EB*, *CF*, and *FD*, taken together; divide (E. 10. 6.) *GH* in *K*, so that *HK* is to *KG* as




*CD* to *AB*; bisect *GK* in *L*, and *HK* in *M*; since, by the construction, *HK* is to *KG* as *CD* to *AB*, *GH* is to *GK*, (E. 18. 5.) as *CD* together with *AB* is to *AB*; but (Constr. and E. 20. 1.) *GH* is greater than *CD* together with *AB*; therefore (E. 14. 5.) *GK* is greater than *AB*; and since *GK* is to *KH* as *AB* to *CD*, and *GK* has been shewn to be greater than *AB*, *KH* (E. 14. 5.) is greater than *CD*. Wherefore, of the three lines *AB*, *GL*, and *LK*, any two are greater than the third; for *GL* is equal to *LK*, and *AB* together with either of them is greater than the other; and *GK*, i. e. *GL* together with *LK*, has been proved to be greater than *AB*. In the same manner it may be shewn, that of the three lines *CD*, *KM*, and *MH* any

two are greater than the third; therefore upon *AB* (E. 22. 1.) describe the isosceles triangle *ANB*, having its sides *AN*, *BN* each equal to *GL* or *LK*; and upon *CD* describe the isosceles triangle *COD*, having its sides *CO*, *OD* each equal to *KM* or *MH*. The two triangles *ANB*, *COD* are similar to each other, and have their perimeters, taken together, equal to the perimeters of the triangles *AEB*, and *CFD*.

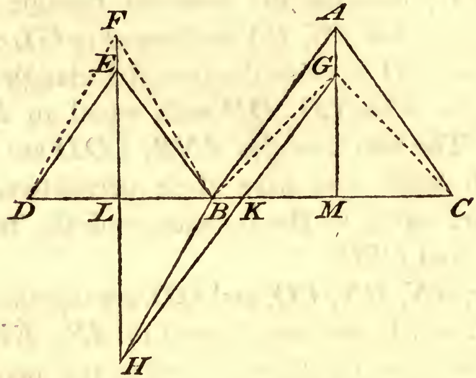
For, *AN*, *BN*, *CO*, and *OD*, are together equal to *GH*, which was made equal to *AE*, *EB*, *CF*, and *FD* taken together; therefore the perimeters of the triangles *ANB*, *COD* are together equal to those of the triangles *AEB*, *CFD*; and because *CD* is to *AB* as *KH* to *GK*, (E. 15. 5.) *CD* is to *AB* as *KM* to *LK*; i.e. *CD* is to *AB* as *CO* to *AN*; wherefore (E. 16. 5.) *CD* is to *CO* as *AB* to *AN*, and (E. 5. 6.) the two triangles *ANB*, and *COD* are similar to each other.

## Prop. XVI.

31. *Theorem.* The two isosceles triangles, standing upon unequal bases, which are similar to each other, are greater, taken together, than the two isosceles triangles together, which stand upon the same bases and have equal perimeters with the two former, but are not similar to each other.

Let *GBC*, *FDB* be two isosceles triangles, not similar to each other, standing upon the bases *BC*, *BD*; of which let *BC* be the greater. Upon the same bases describe (Art. 30.) the two isosceles triangles

*ABC, EDB* of equal perimeter with the former,




and similar to each other; wherefore (E. 19. 6.) the triangle *ABC* is greater than the triangle *EDB*. The two triangles *ABC, EDB* are together greater than the two triangles *GBC, FDB*.

Let the bases *DB*, *BC* be placed in the same straight line; join *F*, *E* and *A*, *G*; and produce *FE* and *AG* to meet *DC* in *L* and *M*; produce *FL* to *H*, and make *LH* equal to *LF*; join *H*, *G* and *H*, *B*; *FD* is equal to *FB*, and *ED* to *EB*, and *FE* is common to the two triangles *DFE*, and *BFE*; therefore (E. 8. 1.) the angle *DFE* is equal to the angle *BFE*; also the angle *FDL* is equal (E. 5. 1.) to the angle *FBL*, and *FL* is common to the two triangles *FLD*, *FLB*; therefore (E. 26. 1.) the triangle *FLD* is equal to the triangle *FLB*, and the angle *FLD* to the angle *FLB*; and (E. 15. and 4. 1.) *FB* is equal to *HB*. In the same manner *EL* may be shewn to bisect the triangle *EDB*, and *AGM* to bisect the triangles *ABC, GBC*.

But, by the hypothesis, $FB$ together with $BG$ is equal to $EB$ together with $BA$; but $FB$ has been proved equal to $HB$; therefore $HB$ together with $BG$ is equal to $EB$ together with $BA$. But (E. 20. 1.) $HB$ together with $BG$ exceeds $HG$; wherefore, also, $EB$ together with $BA$ exceeds $HG$; and (Art. 29.) the square on the aggregate of $EB$ and $BA$ is equal to the square on the aggregate of $EL$, $AM$, together with the square on the aggregate of $LB$, $BM$, that is, to the square on the aggregate of $EL$, $AM$ together with the square on $LM$; therefore these two last squares are together greater than the square on $HG$, i. e. (Art. 29.) greater than the square on the aggregate of $FL$, $GM$, together with the square on $LM$; take away the common square of $LM$, and there remains the square on the aggregate of $EL$, $AM$ greater than the square on the aggregate of $FL$, $GM$; wherefore $EL$ together with $AM$ is greater than $FL$ together with $GM$; take from both these the common parts $EL$, $GM$, and there remains $GA$ greater than $EF$.

And, because the triangle $ABM$ exceeds, by the hypothesis, the similar triangle $EBL$, (E. 10. 6. and Art. 5.) $BM$ exceeds $BL$; wherefore the rectangle contained by $GA$ and $BM$, which (E. 41. 1.) is the double of the triangle $AGB$, exceeds the rectangle contained by $EF$, and $BL$, which is the double of the triangle $FEB$; therefore the triangle $AGB$ exceeds the triangle $FEB$; and if to each of these triangles be added the two triangles

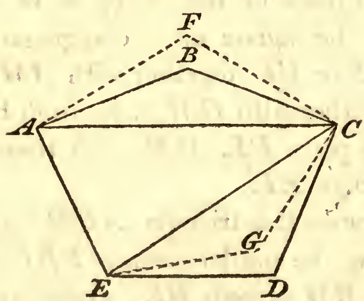
D

*BGM, BEL*, the triangle *AMB* together with the triangle *BEL* will exceed the triangle *FLB* together with the triangle *BGM*; wherefore, also, the doubles of the two former triangles will together exceed the doubles of the two latter taken together; i. e. the triangle *ABC* together with the triangle *EDB* is greater than the triangle *GBC* together with the triangle *FDB* *.

## Prop. XVII.

32. *Theorem.* If a polygon be not equilateral and equiangular, a greater polygon may be found which has an equal perimeter, and the same number of sides.

Let *ABCDE* be any given polygon; if it be not equilateral a greater polygon may be found of




equal perimeter, and the same number of sides, by Art. 9.; let, therefore, *ABCDE* be equilateral

* The sides of the two dissimilar isosceles triangles are, in the figure, taken equal, each to each; that being the case which is used in Art. 32; and the triangle *DEB* then necessarily falls within *DFB*, and *BAC* without *BGC*.

but not equiangular; join $A$, $C$ and $C$, $E$; then, because the angle at $B$ is not equal to the angle at $D$, the two isosceles triangles $ABC$, $CDE$ are not (E. 32. 1.) similar, and (E. 24. 1.) $AC$ and $CE$ are unequal; upon $AC$ and $CE$ describe (Art. 30.) the two similar isosceles triangles $AFC$, $CGE$ having their perimeters, when taken together, equal to the perimeters of the triangles $ABC$, $CDE$, taken together; then (Art. 31.) the two triangles $AFC$, $CGE$, are together greater than the two $ABC$, $CDE$ together; add to both the remaining part of the figure, and the polygon $AFCGE$ is greater than the given polygon $ABCDE$, and it has the same number of sides, and an equal perimeter.

33. Cor. 1. Of all polygons of the same number of sides, and of equal perimeter, if any one be greatest it is that which is equilateral and equiangular.

For, if there be such a maximum, the figure is either equilateral and equiangular or not; but it cannot be a maximum, if its sides be not equal to each other, and also its angles equal to each other; because, in that case, (Art. 32.) a greater figure might be found. If, therefore, the figure be a maximum, it must be equilateral and equiangular.

34. Cor. 2. Of all equal polygons, of the same number of sides, if the perimeter of one be a minimum, it is that of the regular polygon, (Art. 24. and 33.)

## SCHOLIUM.

It is usually asserted, in the place of Art. 32. that an equilateral and equiangular polygon is the greatest of all isoperimetrical polygons of the same number of sides; and the steps by which this property is commonly demonstrated are these:— 1. Of rectilineal plane figures, in which all the sides, but one, are given, the greatest is that which may be inscribed in a semi-circle, having the undetermined side for its diameter. 2. Hence, of all rectilineal plane figures, contained by sides which are given both in number and length, the greatest is that which may be inscribed in a circle. 3. The greatest of all rectilineal plane figures of the same perimeter and number of sides, is that which is equilateral. 4. Therefore the greatest of such figures is that which is equilateral, and which, at the same time, is capable of being inscribed in a circle; i. e. which is equilateral and equiangular. But from the proof given of the first step nothing more can be strictly concluded, than what is expressed in Art. 11.; therefore, the deduction contained in the second step is not fairly drawn. Again, in the third step the same unauthorized extension is made of Art. 17. that is made of Art. 11. in the first step. No more, therefore, is thus proved than what is stated in Art. 32. The advantage of the method of proof here adopted is, that it requires only *one* hypothesis, namely, that

the greatest of all isoperimetrical polygons, of the same number of sides, is necessarily equilateral; whereas the more common method of proof rests upon *two* distinct suppositions.

It may also be remarked, that if two polygons, contained by straight lines given both in length and number, be inscribed in the same, or equal, circles, they shall be equal to each other, whatever be the order of their sides. For the polygon inscribed in a circle is equal to the difference between the circle, and the segments of the circle cut off by its sides; and the aggregate of the segments cut off will manifestly (E. 28. 3. E. 21. 3. and E. 24. 3.) be the same, whatever be the order of the sides of the inscribed figure.

Further, it may be shewn, as is done by LEGENDRE, that, if the sides of a polygon be given both in length and number, the radius of the circle in which a polygon, so bounded, may be inscribed can only be of one certain length. For, if it be possible, let two such polygons be inscribed in two circles having unequal radii; then, if the centers of the circles and each angular point in both figures be joined, the angles at the center will, in both cases, be equal to four right angles; but the angles at the center of the greater circle are together less than the angles at the center of the other; for the straight lines by which they are subtended are equal, each to each; and the same straight line subtends a greater angle (E. 21. 1.) at a less perpendicular distance from its bisection;

wherefore the angles at the centers of the two circles are at the same time equal and unequal, which is absurd.

## Prop. XVIII.

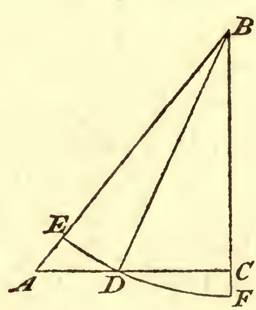
35. *Problem.* To find the center of a circle which may be described about any given regular polygon.

The construction and the proof are the same as in E. 14. 4.

## Prop. XIX.

36. *Theorem.* If a straight line be drawn from either of the acute angles of a right-angled triangle to cut the opposite side, that side shall have to the segment cut off from it, a greater ratio than the whole acute angle has to the part of it cut off, by the straight line.

Let *BCA* be a right-angled triangle, and let




the straight line *BD* be drawn from the acute

angle $B$, cutting the opposite side $AC$ in $D$; $AC$ has a greater ratio to $DC$ than the angle $ABC$ has to the angle $DBC$.

For $DB$ is (E. 17. and 19. 1.) less than $BA$, and greater than $BC$; if, therefore, a circle $EDF$ be described, from $B$ as a center, at the distance $BD$, it will cut $BA$ between $A$ and $B$ in $E$, and $BC$ produced in $F$. The triangle $ABD$ is to the triangle $DBC$ as $AD$ to $DC$ (E. 1. 6.); but the triangle $ABD$ is greater than the sector $EBD$; and the triangle $DBC$ is less than the sector $DBF$; therefore (E. 13. 5.) $AD$ has to $DC$ a greater ratio than the sector $EBD$ has to the sector $DBF$, that is, (E. 33. 6.) a greater ratio than the angle $ABD$ has to the angle $DBF$; and, *componendo* *, $AC$ has to $DC$ a greater ratio than the angle $ABC$ has to the angle $DBC$.

## Prop. XX.

37. *Theorem.* Of regular polygons, which have equal perimeters, that which has the greatest number of angles is the greatest.

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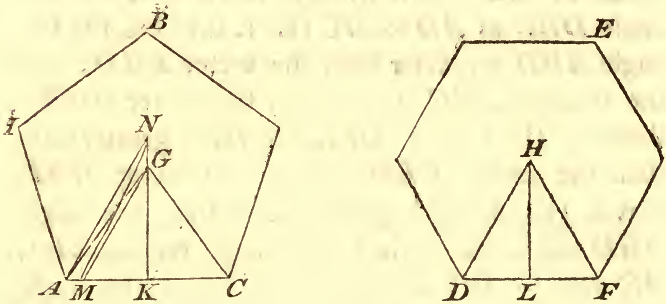
* Euclid cannot be quoted for this step; it may thus be proved. Let $A$ have to $B$ a greater ratio than $C$ has to $D$. Find (E. 12. 6.) $E$ a fourth proportional to $D$, $C$, and $B$, so that $E : B :: C : D$; therefore (E. 10. 5.) $A$ is greater than $E$; add $B$ to both; and $A$ together with $B$ is greater than $E$ together with $B$; therefore (E. 8. 5.) $A$ together with $B$ has a greater ratio to $B$ than $E$ together with $B$ has to $B$; i. e. (E. 18. 5.) than $C$ together with $D$ has to $D$.

Let the two regular polygons *ABC*, *DEF*, have equal perimeters; but let the polygon *DEF* have the greater number of angles; then is it greater than the polygon *ABC*.

Find (Art. 35.) the centers, *G* and *H*, of the circles circumscribing the polygons *ABC*, *DEF*; from *G* and *H* draw (E. 12. 1.) *GK* per-




pendicular to *AC*, and *HL* perpendicular to *DF*; and (E. 3. 3.) they will bisect the angles *AGC*, *DHF*, and the bases *AC*, *DF*; also join *A*, *G* and *G*, *C*, and *D*, *H* and *H*, *F*. Then since the figure *DEF* has (hyp.) more angles than the figure *ABC*, it has also more sides; and since the two perimeters (hyp.) are equal, *DF* is less than *AC*; and, therefore, *DL*, which is the half of *DF*, is less than *AK*, which is the half of *AC*. From *AK* cut off *MK* (E. 3. 1.) equal to *DL*, and join *M*, *G*: *AC* is to the perimeter of the figure *ABC* as the angle *AGC* is to four right angles; because all the angles at the center *G*, subtended by the equal sides of the polygon, are equal to each other;

and are together equal to four right angles; also, the perimeter of the figure $DEF$ is to $DF$ as four right angles are to the angle $DHF$; therefore (E. 22. 5.) *ex æquali*, $AC$ is to $DF$ as the angle $AGC$ to the angle $DHF$; therefore, also, (E. 15. 5.) $AK$ is to $DL$, or $MK$, as the angle $AGK$ to the angle $DHL$. But (Art. 36.) the ratio of $AK$ to $MK$ is greater than the ratio of the angle $AGK$ to the angle $MGK$; wherefore the ratio of the angle $AGK$ to the angle $DHL$, is greater than that of the angle $AGK$ to the angle $MGK$, and (E. 10. 5.) the angle $MGK$ is greater than the angle $DHL$; therefore (E. 32. 1.) the angle $GMK$ is less than the angle $HDL$. Make (E. 23. 1.) the angle $KMN$ equal to the angle $HDL$; then since $MK$ is equal to $DL$, and the angle at $K$ to the angle at $L$, the two triangles $NMK$, $HDL$ are equal (E. 26. 1.) to each other, and $NK$ is equal to $HL$; therefore $HL$ is greater than $GK$; but the polygon $EDF$ is the half of the rectangle (E. 41. 1. and E. 1. 2.) contained by $HL$ and its whole perimeter; for it is equal to the aggregate of all the equal triangles into which the figure is divided by straight lines drawn from $H$ to the angles; in the same manner, the polygon $ABC$ is the half of the rectangle contained by $GK$ and an equal perimeter; and because $HL$ is greater than $GK$, the former of these rectangles is greater than the latter; wherefore, also, the polygon $DEF$ is greater than the polygon $ABC$.

38. Cor. A regular polygon of any given

number of sides is greater than any other polygon of a less number of sides and of equal perimeter, if the regular polygon of a given number of sides, and of a given perimeter, be a maximum, (Art. 37. and 33.)

## Prop. XXI.

39. *Theorem.* Of all equal regular polygons that which has the greatest number of sides has the least perimeter.

Let $A$ and $B$ be two equal regular polygons of which $A$ has the greater number of sides; the perimeter of $A$ is less than that of $B$.

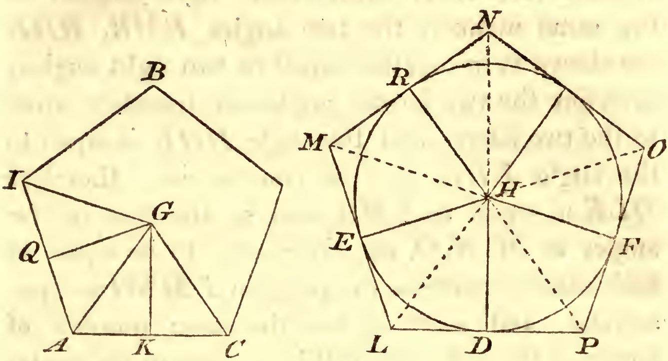
Let a third polygon $C$ be supposed to have the same number of sides as $A$, and the same perimeter as $B$; then $C$ (Art. 37.) is greater than $B$, and, therefore, greater than the equal polygon $A$; its perimeter is, therefore, (Art. 23.) greater than that of $A$; i. e. the perimeter of $B$ is greater than that of $A$.

## Prop. XXII.

40. *Problem.* A regular polygon being given, to describe about a given circle another polygon similar to the given polygon.

Let $ABC$ be the given polygon, and $DEF$ the given circle; it is required to describe about the circle $DEF$ a polygon similar to the given polygon $ABC$.

Find (E. 1.3.) the center $H$ of the given circle; join $H$ and any assumed point $D$, in the circum-




ference; find $G$ (Art. 35.) the center of the circle described about the polygon $ABC$; join $G$, $A$ and $G$, $C$ and $G$, $I$, &c.; then (E. 28. and 27. 3.) the angles at the center $CGA$, $AGI$, &c. are equal to each other; bisect each of these equal angles (E. 9. 1.) by the straight lines $GK$, $GQ$, &c.; at the point $H$ in $HD$ make (E. 23. 1.) the angle $DHE$ equal to the angle $KGQ$, the angle $EHR$ equal to the angle $DHE$, and so on, as often as the polygon has sides; through $D$, $E$, $R$, &c. draw the straight lines (E. 11. 1.) $LP$, $LM$, $MN$, &c. at right angles to $HD$, $HE$, $HR$, &c.; the figure $LMNOP$ is a similar polygon to the given polygon $ABC$, and is circumscribed about the circle $DEF$.

For it is circumscribed about the given circle, because (E. Cor. 16. 3.) each of its sides touches

the circle. And because the angles at $E$ and $D$ are right angles, the two angles $ELD$, $EHD$ are together (E. 32. 1.) equal to two right angles; in the same manner, the two angles $EMR$, $RHE$ are shewn to be together equal to two right angles; therefore the two former angles are together equal to the two latter; and the angle $DHE$ is equal to the angle $EHR$, by the construction; therefore $DLE$ is equal to $EMR$, and so the rest of the angles at $M$, $N$, $O$, &c. are shewn to be equal to each other, therefore the polygon $LMNO$ is equiangular; and since it has the same number of angles as the polygon $ABC$, any one of its angles is equal (E. 32. 1.) to any one of the angles of the polygon $ABC$.

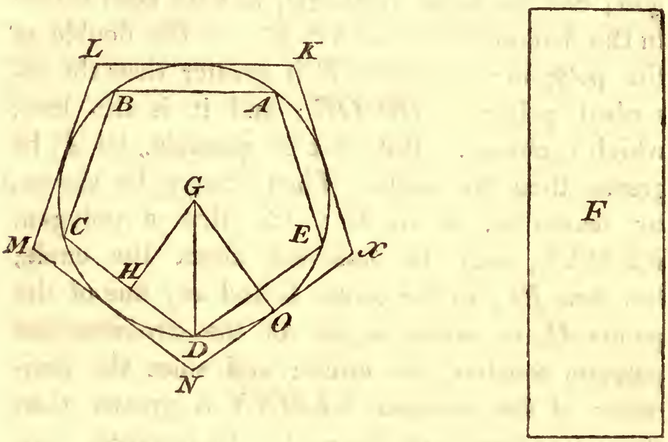
Again, because $LE$ is equal (E. Cor. 36. 3.) to $LD$, and $HE$ to $HD$ (E. Def. 15. 1.) and $LH$ is common to the two triangles $LEH$, $LDH$, the angle $EHL$ is equal to the angle $DHL$; therefore $EHL$ is the half of $EHD$; in the same manner $EHM$ may be shewn to be the half of the angle $EHR$; but the angle $EHR$ is equal to the angle $EHD$; therefore $EHM$ is equal to $EHL$; and the angles at $E$ are right angles, and $HE$ is common to the two triangles $HEM$, $HEL$; therefore $ML$ is the double of $EL$; and in the same manner $LP$ may be shewn to be the double of $LD$; and $LE$ is equal to $LD$; wherefore $ML$ is equal to $LP$; and thus all the sides of the figure may be shewn to be equal to each other; and it has been

already proved to be equiangular; therefore it is similar to the given polygon *ABC*.

## Prop. XXIII.

41. *Theorem.* A circle is equal to the half of the rectangle contained by its circumference and its radius.

Let *ABCD* be the given circle, and let *F* be




the half of the rectangle contained by its circumference and its radius. The circle *ABCD* is equal to the rectangle *F*.

For if it be not equal, it is either greater than it, or less. If it be possible, let *F* be less than the circle; therefore, as is shewn in the demonstration of E. 2. 12. a polygon may be inscribed in the circle, which shall be greater than *F*; let *ABCDE*

be such a polygon, so described, and from the center *G* draw (E. 12. 1.) *GH* perpendicular to any one of its sides, and join *G*, *D*; and, since* the circumference of the circle is greater than the perimeter of the inscribed polygon, and its radius *GD* is greater than *GH*, (E. 17. and 19. 1.) the rectangle contained by the circumference and the radius of the circle is greater than that contained by *GH* and the perimeter of the inscribed polygon; but this latter rectangle, as hath been shewn in the demonstration of Art. 37., is the double of the polygon; therefore *F* is greater than the inscribed polygon *ABCDE*; and it is also less; which is absurd. But, if it be possible, let *F* be greater than the circle. Then it may be shewn, by reasoning, as in E. 2. 12., that a polygon, *KLMNX*, may be described about the circle, less than *F*: join the center *G* and any one of the points *O*, in which a side of the circumscribed polygon touches the circle; and since the perimeter of the polygon *KLMNX* is greater than the circumference of the circle, the rectangle contained by the perimeter of *KLMNX* and *GO*, which is the double of the polygon, is greater than the rectangle contained by the circumference of the circle and *GO*; wherefore the circumscribed

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* It is one of the λαμβανομενα, or axioms, of ARCHIMEDES, that the circumference of a circle is greater than the perimeter of any rectilineal figure inscribed in it; and less than the perimeter of any rectilineal figure described about it.

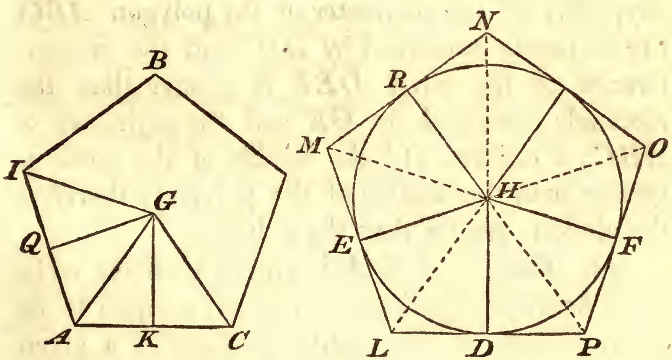
polygon is greater than $F$; and it is also less; which is absurd. Therefore the circle $ABCD$ can neither be greater nor less than $F$; i. e. it is equal to $F$.

42. Cor. The circumferences of circles are to one another as their radii (E. 2. 12. and 22. 6.).

## Prop. XXIV.

43. *Theorem.* A circle is greater than any regular polygon, the perimeter of which is equal to the circumference of the circle.

Let the perimeter of the regular polygon $ABC$ be equal to the circumference of the circle $DEF$;




the circle $DEF$ is greater than the polygon $ABC$.

Find $H$ (E. 1. 3.) the center of the given circle, and (Art. 35.) $G$ the center of a circle described

about $ABC$; about the circle $DEF$ describe (Art. 40.) a polygon $LMNOP$ similar to $ABC$; join $H$, $D$, and from $G$ draw (E. 9. 1.) $GK$ bisecting the angle $AGC$.

Since the perimeter $LMNOP$ is greater than the circumference of the circle $DEF$, it is also greater than the perimeter of the polygon $ABC$, by the hypothesis; and the two polygons have the same number of sides; therefore $LP$ is greater than $AC$; and $LD$, the half of $LP$, is greater than $AK$, the half of $AC$; also the triangle $AGK$, as was shewn in Art. 40., is equiangular with the triangle $LHD$, and, therefore, (E. 4. 6.) $LD : DH :: AK : KG$; wherefore (E. 14. 5.) $DH$ is greater than $KG$; and, because $DH$ is greater than $KG$, and the circumference of the circle is equal (hypoth.) to the perimeter of the polygon $ABC$, the rectangle contained by $DH$, and the circumference of the circle $DEF$ is greater than the rectangle contained by $GK$ and the perimeter of $ABC$; i. e. (Art. 41.) the double of the circle is greater than the double of the polygon; therefore the circle is greater than the polygon.

44. Cor. 1. A circle is greater than any rectilineal figure, the perimeter of which is equal to its circumference, if the regular polygon of a given perimeter and a given number of sides be a maximum (Art. 33.).

45. Cor. 2. The circumference of a circle is less than the perimeter of any equal regular polygon.

This is proved in the same manner as Art. 39.

46. Cor. 3. The circumference of a circle is less than the perimeter of any equal polygon, if the perimeter of the regular polygon of a given area, and a given number of sides, be a minimum, (Art. 34.)

## SCHOLIUM.

It was demonstrated by Galileo that a circle is a mean proportional between any two regular and similar polygons, one of which is isoperimetrical with the circle, and the other circumscribed about the circle. The proposition is equally true, although the two polygons be not regular, provided that they are similar to each other; and hence Art. 43. and 45. may be easily deduced. The relation, also, between the surfaces of isoperimetrical regular polygons which have not the same number of sides, and the relation between their perimeters, when their surfaces are equal, may, by means of this proposition, be collected from the comparison of similar figures, which admit of being circumscribed about the same circle. But, besides that the properties of these circumscribed polygons form a distinct subject, separately investigated in the next section, the method according to which Art. 37. is demonstrated, was preferred, as a specimen of the more ancient Geometry.

If a given quantity of land, supposed to be a plane surface, were required to be enclosed by a

fence of given dimensions, according to any regular figure, with the least quantity of materials, it is manifest from Art. 45. that the fence must be made circular; and it is easy to compute the exact saving which would accrue from the adoption of this form rather than that of other regular figures. If, for example, the quantity of surface to be enclosed be four acres, and $r$ denote the side of a square rood, or 34.78 yards nearly, $O$ the perimeter of an oblong having its sides as 4 to 1, and containing four acres, $T$ the perimeter of an equal equilateral triangle, $S$ that of an equal square, $H$ that of an equal regular hexagon, and $C$ the circumference of an equal circle,

$$O = 20 \times r$$

$$T = 18.236 \times r$$

$$S = 16 \times r; \therefore O - S = 4r = \overset{\text{yards}}{139.12} \text{ nearly.}$$

$$H = 14.889 \times r; \therefore S - H = 1.111r = \overset{\text{yards}}{38.64};$$

$$\text{and } O - H = \overset{\text{yards}}{177.76}.$$

$$C = 14.179 \times r; \therefore S - C = \overset{\text{yards}}{63.33};$$

$$\text{and } H - C = \overset{\text{yards}}{24.69}.$$

The difference between the perimeter of the regular hexagon and that of the equal circle is not, perhaps, so great as, without the calculation, might be expected. But, besides that the advantage is less than might have been conjectured, the circular form could seldom be adopted, unless the surrounding space were waste. Where subdivisions are wanted, the figures which fill space about a

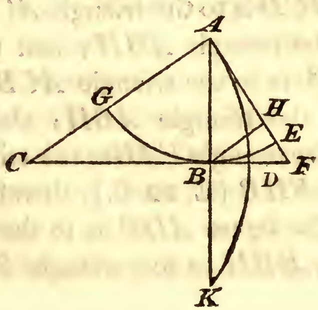
given point must be had recourse to; these, as will be shewn in the third part of this work, are the equilateral triangle, the square, and the regular hexagon; and there is a considerable difference between the lengths of the perimeters of these figures in the above example.

47. Def. An *arch* of a circle is any part of the circumference; a *chord* is any straight line in a circle, terminated both ways by the circumference; a *sagitta* is a straight line joining the bisection of the chord and the bisection of the arch which the chord subtends; and is, therefore, (E. 30. 3. and Cor. 1. 3.) in the circle, a segment of that diameter which is at right angles to the chord.

## Prop. XXV.

48. *Theorem.* Any given sector of a circle has to the trilineal figure contained by its arch, and the semi-chord and sagitta of the double of its arch, a greater ratio than a right angle has to the angle of the given sector.

Let *ACD* be the given sector; also let the arch



*ADK* be the double of the arch *AD*; and let *AK* be the chord, and *BD* the sagitta of the arch *ADK*; the sector *ACD* has to the trilineal figure *ABD* a greater ratio than the right angle *ABC* has to the angle of the sector *ACD*.

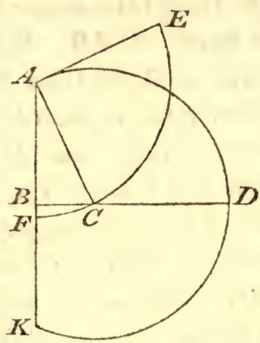
First, let the sector *ACD* be less than a quadrant; draw (E. 11. 1.) *AF* at right angles to *AC*, and *BH* at right angles to *AF*; *AB* is at right angles (Art. 46.) to *CD*; therefore (E. 8. 6.) the triangles *CAF*, *CAB*, *FAB*, *BAH*, *FAH*, are similar to each other; from the center *A* at the distance *AB*, describe the circle *GBE*; the trilineal figure *EBF* has (E. 8. 5.) a greater ratio to the figure *EBH* than to the sector *EAB*; and, therefore, (Art. 36. Note,) *componendo*, the triangle *FBH* has to the figure *EBH* a greater ratio than the triangle *FAB* to the sector *EAB*. But because the angle *ACD* is equal to the angle *BAE* (E. 33. 6. and 11. 5.) the sectors *ACD*, *EAB* are to each other as their respective circles; i. e. (E. 2. 12.) as the square of *AC* to the square of *AB*; and (E. 19. 6.) the two similar triangles *ABC*, *ABH* are to each other in the same ratio; wherefore (E. 11. & 16. 5.) the sector *ACD* is to the triangle *ACB* as the sector *ABE* to the triangle *ABH*; and (E. 17. 5.) the figure *ADB* is to the triangle *ACB* as the figure *BEH* is to the triangle *ABH*; also the triangle *ACB* is to the triangle *FAB* as the triangle *ABH* to the triangle *FHB* (E. 22. 6.); therefore (E. 22. 5.) *ex æquali*, the figure *ADB* is to the triangle *FAB* as the figure *EBH* to the triangle *FHB*; but the

triangle *FHB* was shewn to have a greater ratio to *EBH* than the triangle *FAB* to the sector *EAB*; therefore the triangle *FAB* has a greater ratio to the figure *BAD* than it has to the sector *EAB*; and, therefore, (E. 10. 5.) the sector *EAB* is greater than the trilineal figure *DAB*; *EAB* has, therefore, a greater ratio to *BAG* than *DAB* has; but *DAB* has a greater ratio to *BAG* than it has to *ABC*; much more, then, has *EAB* a greater ratio to *BAG* than *DAB* has to the triangle *BAC*. But the sector *EAB* is to the sector *BAG* as the angle *EAB* to the angle *BAC*; wherefore the angle *FAB* has a greater ratio to the angle *BAC* than the figure *DAB* to the triangle *BAC*; and the triangle *BAC* has to the figure *DAB* a greater ratio than the angle *BAC* to the angle *FAB*; therefore, (Note to Art. 36.) *componendo*, the sector *DCA* has a greater ratio to the figure *DAB* than the angle *FAC* to the angle *FAB*; and *FAC* is a right angle, and *FAB* is equal to the angle of the sector *ACB*; therefore the sector *DCA* has to the trilineal figure *DBA*, a greater ratio than a right angle has to the angle of the sector.

But if the sector *ACD** be greater than a quadrant; draw, as before, *AE* perpendicular to *AC*, and from *A* as a center, at the distance *AC*, describe a circle, and let it meet *AE* in *E*, and *AB* produced in *F*; the angle *ACD* is greater than the angle *CAE*; and the two circles *ADK*, *CEF* are

* See the figure in the following page.

equal; therefore the sector *ACD* is greater than the sector *EAC*; wherefore the sector *ACD* has a greater ratio to the triangle *ABC* than the sector *ACE* has to it; much more then has the sector

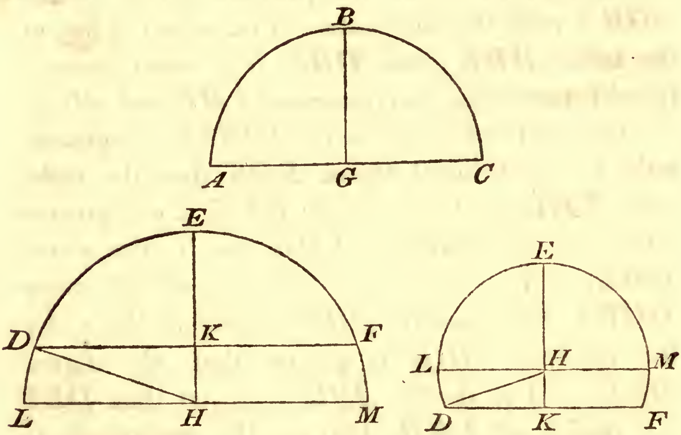



*ACD* a greater ratio to the triangle *ABC* than the sector *ACE* has to the sector *ACF*, which is greater than *ABC*; but (E. 33. 6.) the sector *ACE* is to the sector *ACF* as the angle *EAC* to the angle *CAF*; therefore the sector *ACD* has a greater ratio to the triangle *ABC*, than the angle *EAC* has to the angle *CAF*; and (*convertendo* and *componendo*) the sector *ACD* has a greater ratio to the trilineal figure *ABD*, than the angle *EAC* has to the angle *EAF*; but the angle *EAC* is a right angle, and the angle *EAF* is, therefore, equal to a right angle together with the angle *CAB*; as also (E. 32. 1.) is the exterior angle *ACD*; therefore the angle *EAF* is equal to the angle *ACD*; and the sector *ACD* has a greater ratio to *ABD*, than a right angle has to the angle of the sector *ACD*.

## Prop. XXVI.

49. *Theorem.* Of circular segments, the arches of which are equal, the greatest is the semi-circle.

Let the segment *ABC* be a semi-circle, and let




its arch *ABC* be equal to the arch *DEF* of any other circular segment; the semi-circle *ABC* is greater than the segment *DEF*.

First, let the segment *DEF* be less than a semi-circle; find (E. 1. and 25. 3.) the centers, *G* and *H*, of the two circles of which the two given figures are segments; draw (E. 11. 1.) the straight lines *GB*, at right angles to *AC*, and *HKE*, at right angles to *DF*; through *H* draw (E. 31. 1.) the diameter *LHM* parallel to *DF*, and join *H, D*; then (E. 30. 3.) *AGB* and *LHE* are quadrants, and (Art. 42. and E. 15. 5.) the arch *LDE* is to the

arch *AB* as *HL* to *GA*; but, by the hypothesis, *AB* is equal to the arch *DE*; wherefore *LE* is to *DE* as *HL* to *GA*; but (E. 33. 6.) *LE* is to *DE* as the quadrant *LHE* to the sector *DHE*; and (E. 2. 12. and 15. 5.) the square of *HL* is to the square of *GA* as the quadrant *LHE* to the quadrant *AGB*; wherefore the quadrant *LHE* has to *AGB* a ratio the duplicate of that which it has to the sector *DHE*; and *DHE* is a mean proportional between the two quadrants *LHE* and *AGB*.

But (Art. 48.) the sector *DEH* has a greater ratio to the trilineal figure *EDK* than the right angle *LHE* has to the angle *DHE*, i. e. a greater ratio than the quadrant *LHE* has to the sector *DHE*, and a greater ratio than that of the sector *DHE* to the quadrant *AGB*; therefore (E. 8. 5.) the quadrant *AGB* is greater than the figure *DKE*, and its double *ABC* is greater than *DEF* the double of *DKE*; that is, the semi-circle is greater than the segment which has an equal arch, and which itself is less than the half of the circle to which it belongs.

But if *DEF* be greater than a semi-circle, it may be shewn, as before, that the quadrant *LHE* is to the sector *DHE* as the sector *DHE* is to the quadrant *ABG*; and, therefore, (Art. 48.) the sector *DHE* has a greater ratio to the figure *DKE* than the right angle *LHE* has to the angle *DHE*, i. e. a greater ratio than that of the quadrant *LHE* to the sector *DHE*; and, therefore, greater than that of the sector *DHE* to the

quadrant *AGB*; wherefore the quadrant *AGB* is greater than the figure *DEK*, and its double *ABC*, greater than *DEF* the double of *DEK*.

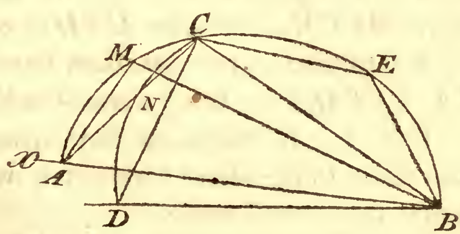
50. Cor. The space which can be enclosed by a given finite line, together with an indefinite straight line, the given line being disposed in the form of a circular arch, is greatest when the arch is a semi-circle.

51. *Postulate.* A plane figure of any kind having been described upon a finite straight line, let it be granted, that a similar and equal figure may be described, or be supposed to be described, upon an equal straight line.

## Prop. XXVII.

52. *Theorem.* If space be to be enclosed by a given finite line, together with an indefinite straight line, and if the enclosed space be not made a semi-circle, a greater space may be found under the same conditions.

Let the figure *ACEB* be contained by the




finite line *ACEB* of given length, and the part

*BA* of the indefinite straight line *BX*; if the figure *ACEB* be not a semi-circle, a greater space may be found, contained by a line of the given length, and part of an indefinite straight line.

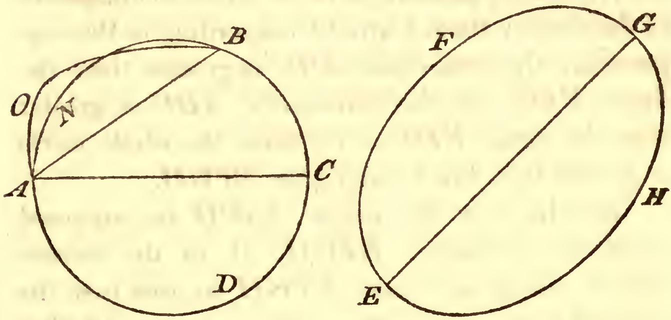
Let *ACEB* be not a semi-circle; if the straight lines drawn from *A* and *B*, to any the same point *C* in the perimeter *ACEB*, are in every such point, at right angles to each other, *ACEB* (E. 31. 1.) is a semi-circle; but it is not, by the hypothesis; therefore there is some point in the perimeter *ACEB*, to which, if straight lines be drawn from *A* and *B*, they will not be at right angles to each other; let *C* be that point; join *A, C* and *B, C*; then, since the angle *BCA* is not a right angle, draw (E. 11. 1.) *CD* at right angles to *CB*, and (E. 3. 1.) make it equal to *CA*; upon *CD* let the figure *CND* be supposed (Art. 51.) to be described, similar and equal to *CMA*; and join *B, D*; then the triangle *BCD* is greater (Art. 8.) than the triangle *BCA*; and if there be added to *BCD* the two figures *BEC, CND*, and to *BCA* the two figures *BEC, CMA*, the whole figure *BECND* is proved to be greater than the whole figure *BECMA*, because *CND* is equal to *CMA*; i. e. a greater figure has been found than *BECMA*, *BECMA* not being a semi-circle.

53. Cor. 1. If there be any space, enclosed according to the above conditions, which is a maximum, it is a semi-circle.

54. Cor. 2. If the semi-circle be the greatest space which can be so enclosed, then the figure

contained by two given finite lines, one of which is straight, is a maximum, when it is a segment of a circle, having the given finite straight line for its chord.

Let the space *ANB* be contained by the finite straight line *AB*, and the finite line *ANB* disposed in the form of an arch of a circle, the re-




maining part of which, *BCDA*, may be found by E. 25. 3.; and let *AOB* be any other figure contained by *AB* and the boundary *AOB* equal to *ANB*; draw the diameter of the circle *AC*; then if the semi-circle *ABC* be a maximum under the above specified conditions, the segment *ANB* is greater than the figure *AOB*.

For then the whole figure *ANBC* is greater than the whole figure *AOBC*; and if from both be taken the common part *ABC*, there remains the segment *ANB* greater than the figure *AOB*.

55. Cor. 3. The same supposition being made, the circle is greater than any other plane figure of equal perimeter; and the circumference

of the circle is less than the perimeter of any other equal plane figure.

First, let the circle *ABCD**, and the plane figure *EFGH* have equal perimeters, the circle is the greater.

For draw any diameter *AC* of the circle; and let *EFG* be taken in the perimeter of the other figure, and supposed equal to the semi-circumference of the circle; then, (Art. 53.) according to the supposition, the semi-circle *ABC* is greater than the figure *EFG*, and the semi-circle *ADC* is greater than the figure *EHG*; wherefore the whole circle is greater than the whole figure *EFGH*.

But let now the circle *ABCD* be supposed equal to the figure *EFGH*; if in the former case it be greater than *EFGH*, in this case its circumference is less than the perimeter of that figure; for it cannot be equal to that perimeter, because then the circle would be the greater figure of the two; neither can it be greater than the perimeter, because (Art. 42. and E. 2. 12.) the circle of which it is the circumference would be still greater, and would not be equal to the figure *EFGH*.

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* See the figure in p. 75.

ON

# MAXIMA AND MINIMA.

## PART I.

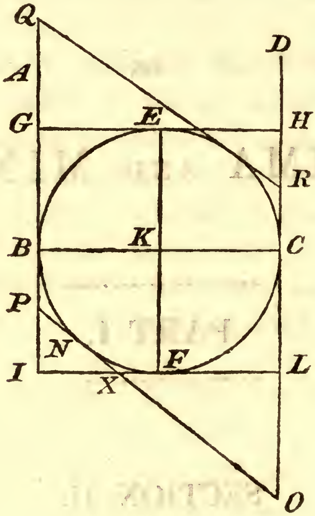
### SECTION II.

PROP. I.

56. *Problem.* IF two given straight lines touch a given circle, to draw the shortest tangent to the same circle, on either side of the straight line joining the given points of contact, which is terminated by the two given tangents.

Let the two straight lines *AB, CD,* which are parallel, or the two *AB, AC* which meet in the point *A,* touch the given circle *BECF* in *B*

and $C$; it is required to draw the shortest line, on either side of the straight line $BC$ joining $B$ and $C$,

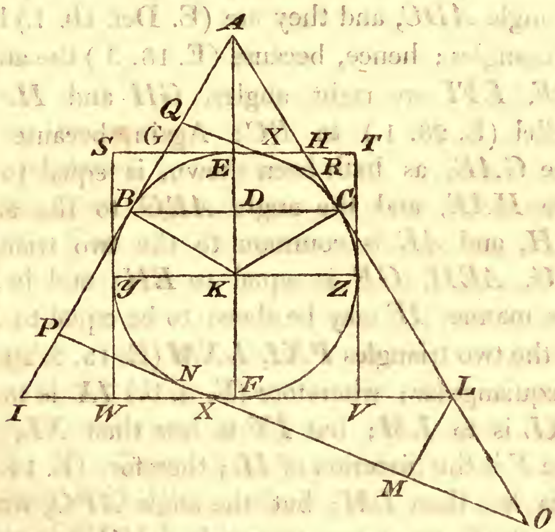



which shall touch the circle, and be terminated by the first two tangents, $AB$, $DC$, or $AB$, $AC$.

First, let the straight lines $AB$, and $CD$ be parallel; find (E. 1. 3.) the center $K$ of the circle $BECF$; draw the diameter (E. 31. 1.) $EKF$ parallel to $AB$, or $DC$, and through the points $E$ and $F$ draw (E. 17. 3.) $GH$, and $IL$ touching the circle; $GH$ is shorter than any other tangent to the circle on the same side of $BC$; and $IL$ shorter than any other tangent on the same side with itself of $BC$, both being terminated by $AB$ and $DC$.

For draw *PNO* touching (E. 17. 3.) the circle in any other point than *F*; and because (E. 18. 3.) the angles at *E* and *F* are right angles, *GE* is parallel (E. 28. 1.) to *IF*, and the figures *GF*, *EL*, *GL* are parallelograms; and, therefore, (E. 34. 1.) the angles at *I* and *L* are right angles; therefore (E. 17. 1.) the angles *XPI*, *XOL* are each of them acute; and (E. 19. 1.) *PX* is greater than *IX*; and *XO* greater than *XL*; wherefore *PX*, together with *XO*, is greater than *IX*, together with *XL*; i. e. *PO* is greater than *IL*.

But let, now, the two given tangents not be parallel, and let them meet in *A*; find (E. 3. 1.)




the center *K* of the circle *BECF*; join *A*, *K*, and let *AK* meet the circumference in *E*, and again,

when produced, in *F*; through *E* and *F* draw (E. 17. 3.) *GH* and *IL* touching the circle; *GH* and *IL* are the shortest tangents on each side of *BC*.

For, join *K*, *B* and *K*, *C*; and let *PNO* and *QR* be any other tangents on each side of *BC*, terminated by *AB* and *AC*, and meeting *GH* and *IL* in the points *X*; draw (E. 31. 1.) *LM* parallel to *AB*. And because (E. 18. 3) the angles at *B* and *C* are right angles, and *KB* is equal to *KC*, and *AB* (E. 36. 3. Cor.) to *AC*; therefore (E. 4. 1.) the angle *BAK* is equal to the angle *CAK*; and *AD* is common to the two triangles *ADB*, *ADC*; wherefore (E. 4. 1.) the angle *ADB* is equal to the angle *ADC*, and they are (E. Def. 10. 1.) both right angles; hence, because (E. 18. 3.) the angles *GEF*, *EFI* are right angles, *GH* and *IL* are parallel (E. 28. 1.) to *BC*: Again, because the angle *GAE*, as hath been shewn, is equal to the angle *HAE*, and the angle *AEG* to the angle *AEH*, and *AE* is common to the two triangles *AEG*, *AEH*, *GE* is equal to *EH*; and in the same manner *IF* may be shewn to be equal to *FL*. But the two triangles *PXI*, *LXM* (E. 15. & 29. 1.) are equiangular; wherefore (E. 4. 6.) *IX* is to *IP* as *XL* is to *LM*; but *IX* is less than *XL*, because *F* is the bisection of *IL*; therefore (E. 14. 5.) *IP* is less than *LM*; but the angle *APO*, which is equal (E. 28. 1.) to the angle *LMO*, is greater (E. 16. 1.) than the angle *AIL*, which is equal to the angle *ALI*; and *ALI* is greater than *LOM*;

wherefore $LMO$ is greater than $LOM$, and (E. 19. 1.) $LO$ is greater than $LM$; much more then is $LO$ greater than $PI$; but $LC$ is equal (E. 36. 3. Cor.) to $LF$, and is, therefore, equal to $IF$; and if to $LO$ and $IP$ be added the equals $LC$ and $IF$, $OC$ will be greater than $FI$ and $IP$; therefore $ON$, which is equal to $OC$, is greater than $FI$ and $IP$; add $PN$ to $ON$, and $PB$ to $FI$ and $IP$, and $OP$ will be greater than $FI$ and $IB$; and, therefore, greater than the double of $IF$, i. e. than $IL$.

And $RQ$, in both the cases, may be proved to be greater than $GH$, in the same manner as $PO$ is proved to be greater than $IL$.

57. Cor. 1. The two shortest tangents, so drawn, are bisected in their respective points of contact, and are perpendicular to the straight line joining the points of their contact; which straight line, produced, passes through the intersection of the two given tangents, when they cut one another.

58. Cor. 2. The perimeter of a triangle described about the given circle, and contained by two given tangents which meet, and the greater of the two shortest tangents, so drawn, is less than the perimeter of any other triangle described about the same circle, and having the same vertical angle.

For $IB$, $IL$, and $LC$ are, together, the double of $IL$; and $PB$, $PO$, and $OC$ are, together, the double of $PO$; but $IL$ has been shewn to be less than $PO$; wherefore $IB$, $IL$, and $LC$ are together less than $PB$, $PO$, and $OC$; add to both $AB$ and

*AC*, and the perimeter of the triangle *AIL* is manifestly less than that of the triangle *APO*.

59. Cor. 3. The perimeter of the quadrilateral rectilineal figure *IGHL*, described about a circle, and contained by any two given tangents, and the two shortest tangents included between them, is less than that of any other such figure *PQRO*, contained by the two given tangents, and any two other tangents which meet them, and are not the shortest.

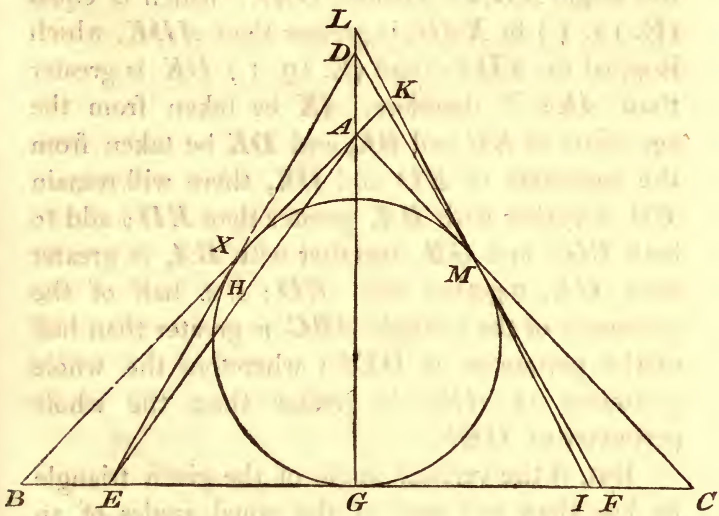
For, *GH* is less than *QR*, and *IL* than *PO*; but *OR*, together with *QP*, is equal to *PO*, together with *QR*, because (E. 36. 3.) *OC* is equal to *ON*, *BP* to *PN*, and *CR* to *RX*; also *IG*, together with *LH*, is, in the same manner, shewn to be equal to *IL* together with *GH*; therefore the whole perimeter of *IGHL* is less than that of *PQRO*.

## Prop. II.

60. *Theorem.* The perimeter of the equilateral triangle described about a circle, is less than that of any other triangle described about the same circle.

Let *ABC* be a triangle described about the circle *GHM*: Either the side *BC* is bisected at right angles in the point of contact *G*, or else (Art. 57. and 58.) another triangle, of less perimeter, may be described about the given circle, having its side opposite to the angle *A* bisected in the point of contact. But let *BC* be so bisected in *G*; and, first, let the angle *BAC* be greater

than any one of the angles of an equilateral triangle. Describe (E. 3. 4.) about the circle *GHM* the equilateral triangle *DEF*, touching the circle in the points *G*, *H*, and *M*. Then, if *A*, *G* be joined by the straight line *AG*, the angle *AGB* is equal (Art. 57.) to the angle *AGC*, and (E. 19. 3.)




the center of the circle is in *AG*; in the same manner, if *D*, *G* be joined, the center of the circle may be shewn to be in *DG*; wherefore *DA* and *AG* are in the same straight line; and since, by the hypothesis, the angle *EDF* is less than the angle *BAC*, the angle *XDA*, which (E. 4. 1.) is the half of *EDF*, is less than the angle *XAG*, which is the half of *BAC*; therefore (E. 16. 1.) *XAG* is the exterior angle, and the point *D* is further from *BC* than the point *A* is. Produce

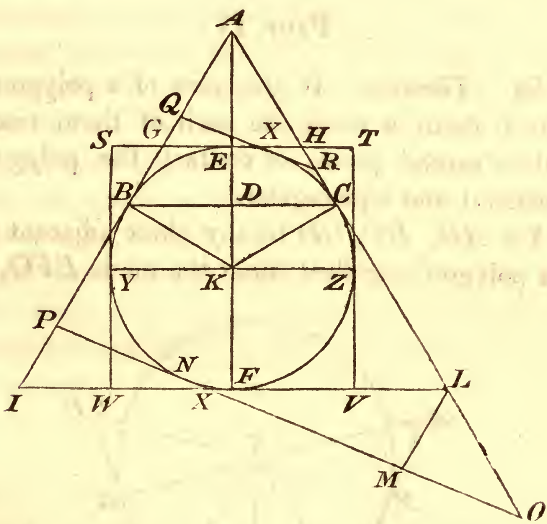
*BA* till it meet *DF* in *K*; then (Art. 58.) the perimeter of the triangle *FBK* is greater than that of *FED*; take from both the common parts *FE*, and *FK*, and there will remain *EB*, together with *BK*, greater than *ED*, together with *DK*; but the angle *XAG* was shewn to be greater than the angle *XDA*; whence *DAK*, which is equal (E. 15. 1.) to *XAG*, is greater than *ADK*, which is equal to *XDA*; and (E. 19. 1.) *DK* is greater than *AK*; if, therefore, *AK* be taken from the aggregate of *EB* and *BK*, and *DK* be taken from the aggregate of *ED* and *DK*, there will remain *EB*, together with *BA*, greater than *ED*; add to both *EG*; and *GB*, together with *BA*, is greater than *GE*, together with *ED*; i. e. half of the perimeter of the triangle *ABC* is greater than half of the perimeter of *DEF*; wherefore the whole perimeter of *ABC* is greater than the whole perimeter of *DEF*.

But, if the vertical angle of the given triangle be less than any one of the equal angles of an equilateral triangle, the angle at its base (E. 5. and 32. 1.) will be greater, because *ABC* is isosceles; therefore (Art. 56. and 58.) another triangle may be described about the circle of less perimeter, and having the angle at the base of the given triangle for its vertical angle; but the perimeter of such a triangle has been shewn to exceed that of an equilateral triangle; much more, then, does the perimeter of the given triangle exceed that of an equilateral triangle.

## Prop. III.

61. *Theorem.* The perimeter of the square described about a given circle, is less than that of any quadrilateral rectilineal figure described about the same circle.

Let *PQRO* be any quadrilateral rectilineal figure described about the circle *BECF*; if *PO*



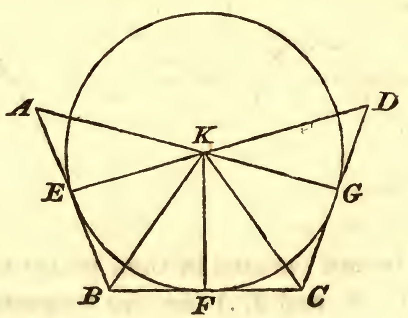

and *QR* be not bisected in their points of contact, draw (Art. 56. and 57.) the two tangents *IL* and *GH*, which are bisected in the points of contact *F* and *E*; and if *GI* and *HL* be not bisected in their points of contact, draw *SW* and *TV*, in the same manner, so as to be bisected in the

points *Y* and *Z*, where they touch the circle; the perimeter of the figure *PQRO* is greater (Art. 59.) than that of the figure *IGHL*; and the perimeter of *IGHL* is greater than that of *WSTV*; the perimeter, therefore, of *PQRO* is greater than that of *WSTV*; and it is evident, from the demonstration of the former part of Art. 56, and Art. 59, that *WSTV* is a square.

## Prop. IV.

62. *Theorem.* If the sides of a polygon described about a circle be each of them bisected in their several points of contact, the polygon is equilateral and equiangular.

Let *AB*, *BC*, *CD* be any three adjacent sides of a polygon described about the circle *EFG*, and




let them be bisected in their points of contact *E*, *F*, and *G*; the polygon *ABCD* is equilateral and equiangular.

Find (E. 1. 3.) the center of the circle *K*, and join *K*, *E*, and *K*, *B*, and *K*, *F*; then (E. 36. 3.) *BE* is equal to *BF*, therefore *BA*, which is the double of *BE*, is equal to *BC*, which is the double of *BF*; and in the same manner *BC* may be shewn to be equal to *CD*, and the polygon to be equilateral. It is also equiangular; for the angles at *F* (E. 18. 3.) are right angles; *BF* is equal to *FC*, and *KF* common to the two triangles *KBF*, *KFC*; therefore (E. 4. 1.) the angle *KBF* is equal to the angle *KCF*; and in a similar manner it may be shewn that the angles *EBK* and *KBF* are equal, as also the angles *KCG* and *KCF*; the angle *EBF* is, therefore, the double of the angle *KBF*, and *FCG* is the double of *KCF*; but the angle *KBF* was shewn to be equal to the angle *KCF*; therefore the angle *ABC* is equal to the angle *BCD*; and in the same manner the remaining angles may be shewn to be equal; therefore the figure is also equiangular, as well as equilateral.

## Prop. V.

63. *Theorem.* If a polygon described about a circle be not equilateral and equiangular, another polygon of the same number of sides may be described about the same circle, which has a less perimeter.

Let *PQRO* (Fig. Art. 61.) be any polygon described about the circle *BECF*; if any of its

sides, *PO*, be not bisected in its point of contact *N*, draw (Art. 56. and 57.) the tangent *IL*, which is bisected in its point of contact *F*; it was shewn, in Art. 56, that *IL* is less than *PO*, and *PI* less than *LO*, and the remaining part of the perimeter is common to the two figures *IQRL*, *PQRO*; therefore the perimeter of *IQRL* is less than that of *PQRO*; and, thus, if every one of the sides of *PQRO* be not bisected in its point of contact, i. e. (Art. 62.) if the figure be not equilateral and equiangular, another polygon, of the same number of sides, may be described about the circle, which has a less perimeter.

## Prop. VI.

64. *Theorem.* Any polygon, which is described about a circle, is equal to the half of the rectangle contained by the perimeter of the polygon, and the radius of the circle.

For it may be divided into as many triangles as it has sides, each of which is equal (E. 41. 1.) to half of the rectangle contained by a side and the radius of the circle; therefore the whole polygon, which is equal to the aggregate of these triangles, is also equal to half of a rectangle, which is the aggregate of the rectangles, i. e. (E. 1. 2.) to half of the rectangle contained by the perimeter of the polygon and the radius of the circle.

65. Cor. All isoperimetrical polygons de-

scribed about the same circle, are equal to each other, whatever be the number of their sides.

## Prop. VII.

66. *Theorem.* Of all triangles described about the same circle, that which is equilateral is the least.

For (Art. 60.) the perimeter of the equilateral triangle is less than that of any other triangle described about the same circle; and, therefore, the rectangle contained by the perimeter of the equilateral triangle and the radius of the circle, is less than that contained by the perimeter of any other triangle and the same radius; i. e. the double (Art. 64.) of the circumscribed equilateral triangle is less than the double of any other circumscribed triangle; therefore the equilateral triangle is less than any other triangle described about the same circle.

## SCHOLIUM.

This proposition may be thus demonstrated, independently of Art. 60.

First, an isosceles triangle $AIL$ (Fig. Art. 61.) is less than any scalene triangle $APO$, having the same vertical angle, and circumscribed about the same circle. For $LM$ being drawn parallel to $AI$ the two triangles $PXI$, and $LXM$ (E. 15. and 29. 1.) are similar; wherefore (E. 19. 6.) $PXI$ :

*LXM* as the square of *IX* is to the square of *XL*; but *IX* is less than *XL*; wherefore the triangle *PXI* is less than the triangle *LXM*; much more then is it less than the triangle *LXO*; the trapezium *APXL*, together with the triangle *PXI*, is, therefore, less than the same trapezium, together with the triangle *LXO*; i. e. the isosceles triangle *AIL* is less than the scalene triangle *APO*, having the common vertical angle *IAL* *.

Secondly, an equilateral triangle, described about a given circle, is less than any other triangle described about the same circle. For if the circumscribed triangle, which is not equilateral, be not isosceles, a less may be found, by the first case, which is isosceles; let it, therefore, be isosceles as *ABC* (Fig. Art. 60.), and let *DEF* be an equilateral triangle described about the same circle and touching it in the same point *G* as the base of the isosceles triangle; produce *BA* to meet *DF* in *K*. Then, by the former case, the triangle *EDF* is less than the triangle *BKF*; and since the part of *KF*, above the line *AC*, is less than the part below it, it may be shewn, (as in the former case *PXI* was proved to be less than *LXO*) that the triangle *BKF* is less than the triangle *BAC*; much more then is the triangle *DEF* less than the triangle *BAC* †.

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* The demonstration of this case was given by Dr. STEDMAN, (*Phil. Trans.* of 1775.)

† This proposition being so proved, it is evident that Art. 66. might fairly be deduced from it.

### Prop. VIII.

67. *Theorem.* The square is the least of all quadrilateral rectilineal figures described about a given circle.

This is deduced from Art. 61. and 64, in the same manner as Art. 66. was made to follow from Art. 60. and 64.

### Prop. IX.

68. *Theorem.* If a polygon, described about a given circle, be not equilateral and equiangular, a less polygon, of the same number of sides, may be described about the same circle.

This is manifest from Art. 63. and 64.

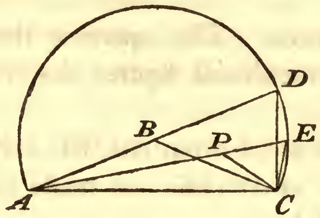
### Prop. X.

69. *Theorem.* Of all triangles, standing upon the same base, and on the same side of it, and having equal vertical angles, the perimeter of that which is isosceles is the greatest.

Let $ABC$ be an isosceles triangle, and $APC$ any other triangle standing upon the same base $AC$, and having its vertical angle $APC$ equal to the vertical angle $ABC$. The perimeter of $ABC$ is greater than that of $APC$.

From the center $B$, at the distance $BA$ or $BC$,

describe the circle *ADC*; produce *AB* and *AP* to meet its circumference in *D* and *E*; and join




*D*, *C* and *E*, *C*; the angle *ADC* is equal (E. 21. 3.) to the angle *AEC*; and *ABC* is equal to *APC* by the hypothesis; but (E. 20. 3.) *ABC* is the double of *ADC*; wherefore, also, *APC* is the double of *AEC*; again, the exterior angle *APC* is equal (E. 32. 1.) to the two interior angles *PEC*, *PCE*; and *APC* is the double of *AEC*; if, therefore, *PEC* be taken from both those equals, there will remain *PEC* equal to *PCE*; therefore (E. 6. 1.) *PE* is equal to *PC*, and *AE* to *AP* and *PC*; also *DB* is equal to *BC*, and *AD* to *AB* and *BC*; but (E. 15. 3.) the diameter *AD* is greater than *AE*; wherefore *AB* and *BC* are, together, greater than *AP* and *PC* together; and if to both *AC* be added, the whole perimeter *ABC* is shewn to be greater than that of *APC**.

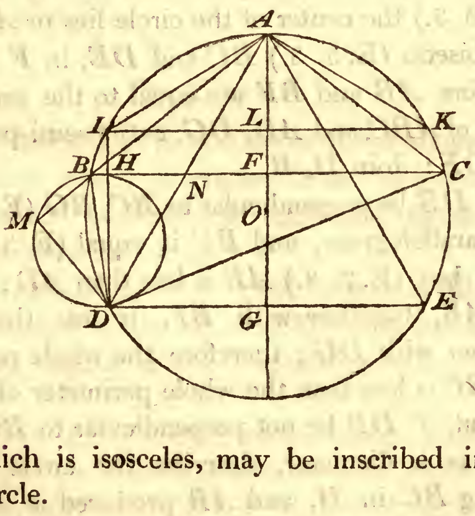
* This Proposition is deduced by Dr. Horsley from the 97th of Euclid's data. In order to make use of Euclid's proposition, it is only necessary to complete the circle *ADC*, bisect the arch which lies below *AC*, and to join the points of the bisection and the points *B* and *P*.

70. Cor. If a polygon, inscribed in a circle, be not equilateral, (and, therefore, also equiangular) another polygon, of the same number of sides, may be inscribed in the same circle which has a greater perimeter.

## Prop. XI.

71. *Theorem.* The perimeter of an equilateral triangle inscribed in a circle, is greater than that of any other triangle inscribed in the same circle.

Let *ABC* be any triangle inscribed in the circle *ABC*. Either *ABC* is isosceles, or else (Art. 69.) another triangle of greater perimeter,




and which is isosceles, may be inscribed in the same circle.

Let, therefore, *ABC* be isosceles; and either its

vertical angle $BAC$ is greater than any one of the angles of an equilateral triangle, or else, as was shewn in the latter part of Art. 60, another isosceles triangle may be inscribed in the circle, having its vertical angle greater than the angle of an equilateral triangle; and (Art. 69.) its perimeter will be greater than that of the former isosceles triangle. Let then $ABC$ be isosceles, and have its vertical angle $BAC$ greater than that of an equilateral triangle. Inscribe (E. 2. 4.) the equilateral triangle $ADE$ in the circle $ABC$, having its base $DE$ parallel to $BC$, and draw $AG$ (E. 12. 1.) perpendicular to $DE$; then (E. 29. 1.) it will be also perpendicular to $BC$; and, because $DE$ is parallel (E. 2. 4. and 27. 1.) to the tangent at $A$, $AG$ is perpendicular to it; therefore (E. 10. 3.) the center of the circle lies in $AG$, and $AG$ bisects (E. 3. 3.) $BC$ and $DE$, in $F$ and $G$; therefore $AB$ and $BF$ are equal to the semi-perimeter of $ABC$, and $AD$, $DG$, to the semi-perimeter of $ADE$: Join $D$, $B$.

If $DB$ be perpendicular to $BC$, $BG$ (E. 29. 1.) is a parallelogram, and $BF$ is equal (E. 34. 1.) to $DG$; but (E. 7. 3.) $AB$ is less than $AD$; wherefore $AB$, together with $BF$, is less than $AD$, together with $DG$; therefore the whole perimeter of $ABC$ is less than the whole perimeter of $ADE$.

But, if $DB$ be not perpendicular to $BC$, upon $DB$, as a diameter, describe the circle $BHD$, cutting $BC$ in $H$, and $AB$ produced in $M$; join $D$, $H$ and $D$, $M$; then since (E. 31. 3.) the angle

*BHD* is a right angle, *HG* is a parallelogram, and *HF* is equal to *DG*; also join *D*, *C*; the angle *MBD* is equal (E. 13. 1. and 22. 3.) to the angle *AED*, and (E. 21. 3.) the angle *DBH* to the angle *DAC*; but *DAC* is greater than *DAE*, or *AED*; therefore the angle *DBH* is greater than the angle *DBM*, and (E. 7. 3.) *MB* is greater than *BH*: Again, because the angle *AMD* is a right angle, *AD* is greater (E. 17. and 19. 1.) than *AM*, i. e. than *AB* and *BM*; much more, then, is *AD* greater than *AB* and *BH*; and if to *AD* be added *DG*, and to *AB* and *BH* be added *HF*, which is equal to *DG*, *AD*, *DG* will be greater than *AB*, *BF*; i. e. the semi-perimeter of *ADE* is greater than the semi-perimeter of *ABC*; and, therefore, the whole perimeter of *ADC* is greater than the whole perimeter of *ABC*.

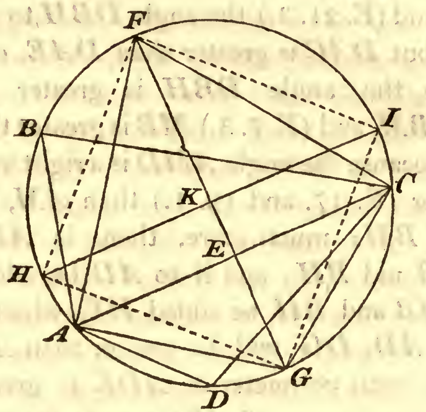
## Prop. XII.

72. *Theorem.* The perimeter of the square inscribed in a circle, is greater than the perimeter of any other quadrilateral rectilineal figure inscribed in the same circle.

Let *ABCD* be any quadrilateral rectilineal figure inscribed in the circle *HFIG*; its perimeter is less than that of a square inscribed in the same circle.

For, join *A*, *C*; and (E. 10. 1.) bisect *AC* in *E*; and (E. 11. 1.) draw the straight line *FEG* perpendicular to *AC*, which will, therefore, (E. 1. 3.) pass through the center, and be a diameter of the

circle; join $A$, $F$ and $F$, $C$, and $A$, $G$ and $G$, $C$; then (E. 4. 1.) the two triangles $AFC$, $AGC$ are

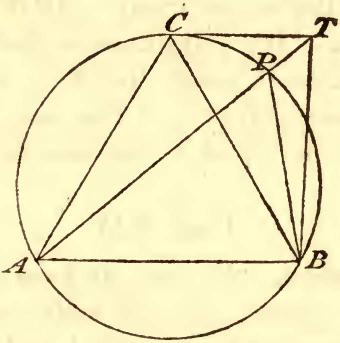



isosceles; and, therefore, (Art. 69.) the perimeter of the figure $AFCG$ is greater than that of the figure $ABCD$; again, bisect the diameter $FG$ in $K$, and draw the diameter $HKI$ at right angles to it; also join $H$, $F$ and $F$, $I$, and $I$, $G$ and $G$, $H$; and it may be shewn, in the same manner, that the perimeter of $HFIG$ is greater than that of $AFCG$; much more, then, is it greater than that of $ABCD$; and (E. 6. 4.) $HFIG$ is a square inscribed in the circle $HFIG$; therefore the perimeter of the inscribed square is greater than that of any other quadrilateral rectilineal figure inscribed in the same circle.

## Prop. XIII.

73. Of all triangles standing upon the same base, and on the same side of it, and having equal vertical angles, that which is isosceles is the greatest.

Let $ACB$ be an isosceles triangle, and $APB$ any other triangle standing upon the same base,



and having its vertical angle $APB$ equal to the vertical angle $ACB$; the triangle $ACB$ is greater than the triangle $APB$.

About the triangle $ACB$ describe (E. 5. 4.) the circle $ACB$; and because the angle $ACB$ is equal to the angle $APB$, the circumference of the circle shall pass through $P$ (E. 21. 3.); otherwise, the exterior angle of a triangle would be equal to the interior opposite angle, which (E. 16. 1.) is absurd.

From the point $C$ draw (E. 17. 1.) the straight line $CT$, touching the circle $ACB$; then (E. 32. 3.) the angle $TCB$ is equal to the angle $CAB$, and,

therefore, (E. 5. 1.) equal to the angle $CBA$; wherefore (E. 27. 1.) $CT$ is parallel to $AB$; and $AP$ produced will meet $CT$; let it be produced to meet $CT$ in $T$, and join $TB$; therefore (E. 37. 1.) the triangle $ACB$ is equal to the triangle $ATB$; but the triangle $ATB$ is greater than the triangle $APB$, which is a part of it; wherefore, also, the isosceles triangle $ACB$ is greater than the scalene triangle $APB$.

74. Cor. If a polygon, inscribed in a circle, be not equilateral, (and, therefore, also equiangular) a greater polygon, of the same number of sides, may be inscribed in the same circle.

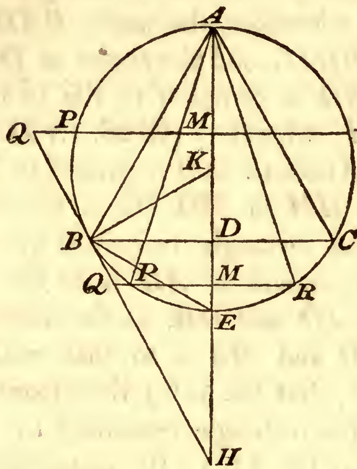
## Prop. XIV.

75. *Problem.* To cut off from the circumference of a given circle an arch, such that the rectangle contained by its chord and sagitta shall be a maximum.

Let $ABC$ be the given circle; it is required to cut off an arch from the circumference of $ABC$, such that the rectangle contained by its chord and sagitta shall be a maximum.

Find (E. 3. 1.) the center $K$ of the given circle, and draw any diameter $AKE$; bisect (E. 10. 1.) $KE$ in $D$, and through $D$ draw (E. 11. 1.) the chord $BDC$ perpendicular to $AE$; the rectangle contained by the chord $BC$ and $DA$, the sagitta of the arch $BAC$ is greater than the rectangle contained by the chord and sagitta of any other arch.

For, take any other point $P$ in the circumference




and draw the chord (E. 12. 1.) $PMR$ perpendicular to $AE$, and (E. 29. 1.) it is parallel to $BC$; join $A$, $B$, and $K$, $B$ and $B$, $E$; also draw (E. 17. 3.) $BH$ touching the circle in $B$, and let $BH$ meet $RP$, produced, in $Q$, and $AE$, produced, in $H$. The two right-angled triangles $BDK$, $BDE$ have the side $BD$ common, and the two other sides $DK$, $DE$, which are about the right angles in each, equal; therefore (E. 4. 1.) $BK$ is equal to $BE$, and $BK$ is also (E. 15. Def. 1.) equal to $KE$; wherefore the triangle $BKE$ is equilateral and (E. 5. 1. Cor.) equiangular.

Again, because (E. 31. 3.) the angle $ABE$ is a right angle, and $BD$ is perpendicular to $AE$, the angle $BAD$ is equal (E. 8. 6.) to the angle $DBE$; and in the same manner $KBD$ may be shewn to

be equal to *DHB*; but *KBD* is equal to *EBD*, the two triangles *BDK*, *BDE* having been proved to be equal; wherefore the angle *BAD* is equal to the angle *BHD*; and the angles at *D* are right angles, and *BD* is common to the two triangles *BDA*, *BDH*; wherefore (E. 26. 1.) *DA* is equal to *DH*. And because *QM* is parallel to *BD*, *HD* is to *DB* as *HM* to *MQ* (E. 4. 6.); wherefore (E. 22. 6.) the rectangle contained by *HD* and *AD*, i. e. the square of *AD* is to the rectangle contained by *AD* and *DB*, as the rectangle contained by *HM* and *MA* is to that contained by *MQ* and *MA*; but (E. 5. 2.) the square of *AD* is greater than the rectangle contained by *HM* and *MA*; therefore (E. 14. 5.) the rectangle contained by *AD* and *DB* is greater than the rectangle contained by *AM* and *MQ*; much more, then, is it greater than that contained by *AM* and *PM*; and, therefore, the double of the rectangle contained by *AD* and *DB*, i. e. the rectangle contained by *AD* and *BC*, is greater than the double of the rectangle contained by *AM* and *PM*, i. e. than the rectangle contained by *AM* and *PR**.

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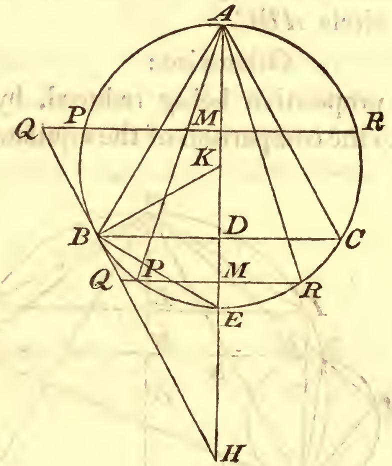
* Since the area of a parabola is varied as the rectangle contained by its axis and terminating ordinate, and since the axes of different parabolas, cut out of the same cone, are varied as the sagittas of the arches cut off, by the planes of the parabolas, from the circular base of the cone, it is manifest, from this proposition, that the greatest parabola, which can be cut out of a given cone, is that, the plane of which divides the diameter of the base of the cone into two parts, which are to each other as 3 to 1.

## Prop. XV.

76. *Theorem.* Of all triangles inscribed in the same given circle, that which is equilateral is the greatest.

Let *ABC* be the given circle, and *APR* any

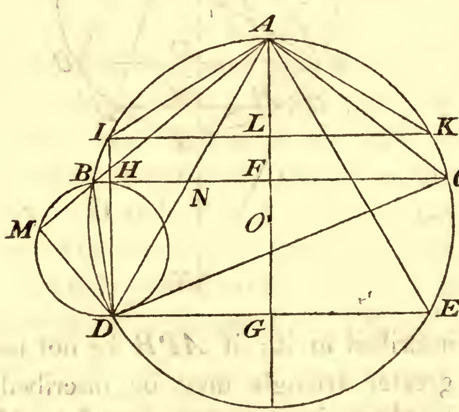



triangle inscribed in it; if *APR* be not isosceles, another greater triangle may be inscribed in it (Art. 73.) which is isosceles; but let *APR* be isosceles; then, the construction of Art. 75. being made, the rectangle contained by *AD* and *BC* is greater than that contained by *AM* and *PR*; and, therefore, (E. 41. 1.) the triangle *ABC* is greater than the triangle *APR*. But the angle *HBD* was shewn (Art. 75.) to be equal to the

angle $BKD$ which is one of the angles of the equilateral triangle $BKE$; and (E. 32. 3.) $HBD$ is equal to the angle $BAC$; wherefore $BAC$ is equal to the angle of an equilateral triangle; and $ABD$ was shewn (Art. 75.) to be equal to $HBD$, and is, therefore, also equal to $BAC$; and (E. 32. 1.) the triangle $ABC$ is equiangular, and, therefore, (E. 6. 1. Cor.) equilateral; and it has been shewn to be greater than any other triangle inscribed in the same circle $ABC$.

Otherwise:

The proposition being reduced, by means of Art. 73, to the comparison of the equilateral triangle




$ADE$ and the isosceles triangle $ABC$ inscribed in the same circle, and $D$, $C$ being joined, the two triangles $ANB$ and $DNC$ are (E. 15. 1. and 22. 3.) similar to each other; wherefore (E. 19. 6.) $DNC$ is to $ANB$ as the square of $DC$ to the square of $AB$; but the angle $DAC$ is greater, and the angle $ADC$ is less,

than an angle of the inscribed equilateral triangle; therefore $DC$ is greater than $AC$ (E. 19. 1.) or than, its equal, $AB$; the triangle $DNC$ is therefore, greater than the triangle $ANB$; add to both the triangle $ANC$, and the whole triangle $ADC$ is greater than the whole triangle $ABC$; but (Art. 73.) the triangle $ADE$ is greater than $ADC$; much more, then, is $ADE$ greater than $ABC$.

## Prop. XVI.

77. *Theorem.* The square inscribed in a circle is greater than any other quadrilateral rectilineal figure inscribed in the same circle.

Let $ABCD$ be any quadrilateral rectilineal

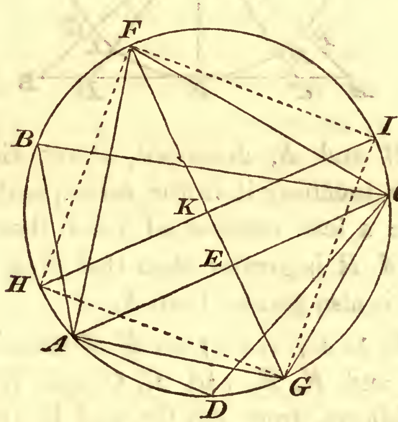
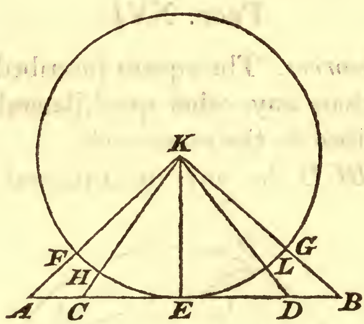



figure inscribed in the circle $HFIG$. It may be shewn to be less than the square $HFIG$ inscribed in that circle, by the help of Art. 73, in the same manner as Art. 72. was deduced from Art. 69.

## Prop. XVII.

78. *Theorem.* **Of two regular polygons described about a circle, that which has the less number of sides has the greater perimeter, and is the greater.**

Let $AB$ and $CD$ be the sides of two regular




polygons $R$ and $S$, described about the circle $FEG$, both touching it in the same point $E$; and let $R$ have a less number of sides than $S$; the perimeter of $R$ is greater than that of $S$, and the polygon $R$ is also greater than $S$.

Find (E. 1. 3.) the center $K$ of the circle, and join $K, A$ and $K, B$, and $K, C$ and $K, D$, and $K, E$; it follows, from Art 62, and E. 18. 3. and 29. 1, that $KE$ bisects the angles $AKB$ and $CKD$; and because the regular polygon $R$ has a less number of sides than $S$, it has also a less number of equal angles, at the center $K$, subtended by its

sides; wherefore (E. 15. 1. Cor. 2.) each of those angles is greater than each of the equal angles subtended by the sides of the other polygon $S$; i. e. the angle $AKB$ is greater than $CKD$; and, therefore, $AKE$ is greater than $CKE$, and $AE$ greater than $CE$, and the side of $R$ greater than that of $S$; and (Art. 36.) the ratio of $AE$ to $CE$ exceeds that of the angle $AKE$ to $CKE$; therefore (E. 15. 5.) the ratio of $AB$ to $CD$ exceeds that of $AKB$ to $CKD$, or (E. 33. 6.) that of the arch $FEG$ to the arch $HEL$; therefore *, also, $AB$ has to $FEG$ a greater ratio than $CD$ has to $HEL$, and (E. 15. and 11. 5.) the perimeter of $R$ has a greater ratio to the circumference of the circle than the perimeter of $S$ has to the same circumference; because the perimeter of $R$ is the same multiple of $CAB$ that the circumference is of $FEG$; and the perimeter of $S$ is the same multiple of $CD$ that the circumference is of $HEL$; wherefore (E. 8. 5.) the perimeter of $R$ is greater than that of $S$; and, it is manifest, also, from Art. 64, that $R$ is greater than $S$.

79. Cor. An equilateral triangle is the greatest of all regular figures described about the same circle, and has the greatest perimeter.

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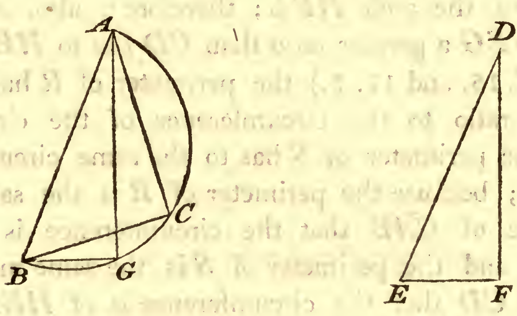
* Let $A : B \sqsubset C : D$; then $A : C \sqsubset B : D$. For, find (E. 13. 6.) $E$ a fourth proportional to $D$, $C$, and $B$; then, because $E : B :: C : D$, and $A : B \sqsubset C : D$, $A$ has a greater ratio to $B$ than $D$ has; and, therefore, (E. 10. 5.) $A \sqsubset E$ and $A : C \sqsubset E : C$, i. e. (E. 16. 5.) $\sqsubset B : D$.

## Prop. XVIII.

80. *Theorem.* If the hypotenuses of two dissimilar right-angled triangles be equal, that triangle which has the less acute angle adjacent to its base, shall have the greater base.

Let $ABC$, $DEF$ be two dissimilar right-angled triangles, having the hypotenuses $AB$, $DE$ equal



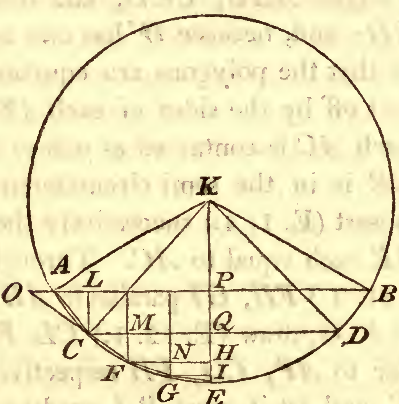
to each other, but the angle $ABC$ less than the angle $DEF$; the base $BC$ is greater than the base $EF$.

Upon $AB$, as a diameter, describe the circle $AGB$; at the point $B$, in $AB$, make the angle $ABG$ (E. 23. 1.) equal to $DEF$, and join $A$, $G$; then because (E. 31. 3.) the angle $AGB$ is a right angle, and $ABG$ equal to $DEF$, and $AB$ to $DE$, $BG$ is equal (E. 26. 1.) to $EF$; but (E. 7. 3.) $BC$ is greater than $BG$; wherefore, also, $BC$ is greater than $EF$.

## PROP. XIX.

81. *Theorem.* Of two regular polygons inscribed in a circle, that which has the greater number of sides has the greater perimeter, and is the greater.

Let $AB$ and $CD$ be the sides of the regular




polygons $V$ and $W$, inscribed in the circle $ABE$; and let $W$ have a greater number of sides than $V$; the perimeter of $W$ is greater than that of $V$, and $W$ is greater than $V$.

First, let $W$ have one more side than $V$. Find (E. 1. 3.) the center $K$ of the circle, and join $K$, $A$, and $K$, $B$, and $K$, $C$, and $K$, $D$; also draw (E. 12. 1.) the radius $KPQE$ at right angles to $AB$ or $CD$, which are supposed to be placed

parallel to each other. And, because $W$ has one more side than $V$, there is one more angle at the center $K$ subtended by the sides of $W$; therefore (E. 15. 1. Cor. 2.) each of the equal angles at the center subtended by the sides of $W$, is less than each of the equal angles there subtended by the sides of $V$; i. e. the angle $CKD$ is less than the angle $AKB$; also, since (E. 3. 3.) $KE$ bisects $AB$ and $CD$ at right angles, it also (E. 4. 1.) bisects the angles $AKB$, $CKD$, and the arches $AEB$, $CED$; and, because $W$ has one more side than $V$, and that the polygons are equilateral, and the arches cut off by the sides of each (E. 28. 3.) equal, the arch $AC$ is contained as many times in $CE$, as $ACE$ is in the semi-circumference; join $A$, $C$, and insert (E. 1. 4.) successively the chords $CF$, $FG$, $GE$ each equal to $AC$. Through $F$ and $G$ draw (E. 31. 1.) $FH$, $GI$ parallel to $AB$ or $CD$; and from $C$, $F$, $G$, draw (E. 12. 1.) $CL$, $FM$, $GN$ perpendicular to $AP$, $CQ$, $FH$ respectively; also produce $FC$, and let it meet $PA$ produced in $O$. Then (E. 16. 1.) the angle $LAC$ is greater than the angle $AOC$; and (E. 29. 1.) the angle $AOC$ is equal to the angle $MCF$; therefore $LAC$ is greater than $MCF$; but $AC$ is equal to $CF$; and (Art. 80.) $CM$ is greater than $AL$; wherefore (E. 8. 5.) $AC$ has a greater ratio to $AL$ than $FC$ to $CM$; and in the same manner $CF$ may be shewn to have a greater ratio to $CM$ than $FG$ to $FN$; and $FG$ to have a greater ratio to $FN$ than

*GE* to *GI*; wherefore* *AC*, *CF*, *FG*, and *GE*, together, have a greater ratio to *AL*, *CM*, *FN*, and *GI*, together, than *CF*, *FG*, *GE*, together, have to *CM*, *FN*, and *GI* taken together; and (Note to Art. 78.) *AC*, *CF*, *FG*, and *GE*, together, have a greater ratio to *CF*, *FG*, and *GE* together, than *AL*, *CM*, *FN*, and *GI* together, have to *CM*, *FN*, and *GI* together; but (E. 28. 3.) the arches *AC*, *CF*, *FG*, and *GE* are equal to each other, because their chords are equal; and, therefore, the arch *ACE* has to the arch *CFE* the same ratio as the aggregate of the chords *AC*, *CF*, *FG*, *GE* to the aggregate of the chords *CF*, *FG*, *GE*; and *AL*, *CM*, *FN*, and *GI* are together (E. 34. 1.) equal to *AP*; and *CM*, *FN*, and *GI* are together equal to *CQ*; wherefore the arch

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* The proposition, here assumed as true, has been proved, in Euclid's manner, by the more ancient Commentators upon his Elements. It may be thus, for the sake of conciseness, demonstrated algebraically.

Let $a : b \sqsubset c : d$, and $c : d \sqsubset e : f$, &c.; then is $a + c + e : b + d + f \sqsubset c + e : d + f$.

For, because $a : b \sqsubset c : d$, and $c : d \sqsubset e : f$, $\frac{a}{b} \sqsubset \frac{c}{d}$, and $\frac{c}{d} \sqsubset \frac{e}{f}$; $\therefore a \,.\, d \sqsubset c \,.\, b$ and $a \,.\, f \sqsubset e \,.\, b$; $\therefore a \,.\, d + a \,.\, f \sqsubset c \,.\, b + e \,.\, b$; add to both $c \,.\, d + e \,.\, d + c \,.\, f + e \,.\, f$; and $a \,.\, d + c \,.\, d + e \,.\, d + a \,.\, f + c \,.\, f + e \,.\, f \sqsubset b \,.\, c + d \,.\, c + f \,.\, c + b \,.\, e + d \,.\, e + f \,.\, e$; i. e. $\overline{a + c + e} \,.\, \overline{d + f} \sqsubset \overline{b + d + f} \,.\, \overline{c + e}$; $\therefore \frac{a + c + e}{b + d + f} \sqsubset \frac{c + e}{d + f}$ and $a + c + e : b + d + f \sqsubset c + e : d + f$.

*ACE* has to the arch *CFE** a greater ratio than *AP* has to *CQ*; and the arch *AEB* has to the arch *CED* a greater ratio than *AB* has to *CD*, each being in this case doubled; therefore, also, the arch *AEB* has to the chord *AB* a greater ratio, than the arch *CED* to the chord *CD*; and the perimeter of *V* is the same multiple of *AB* that the circumference of the circle is of *AEB*; and the perimeter of *W* is the same multiple of *CD* that the circumference is of *CED*; wherefore the circumference has a greater ratio to the perimeter of *V* than to that of *W*; and the perimeter of *V* is, therefore, less (E. 8. 5.) than that of *W*.

Again, it is manifest, by reasoning as in Art. 64, that *W* is equal to half of the rectangle contained by its perimeter and *KQ*, and that *V* is equal to half of the rectangle contained by its perimeter and *KP*, which is less than *KQ*; hence *W* is plainly greater than *V*.

Let, now, *X* be a regular polygon inscribed in the same circle, and having one more side than *W*; it may be shewn in the same manner to be greater than *W*, and to have a greater perimeter; therefore it is greater, also, and has a greater perimeter, than *V*; and thus the proposition may be proved

* Whatever be the curve *ACE*, it has been shewn by Barrow that this property obtains; the greater arch *ACE* has always a greater ratio to the less arch *CE*, than the ordinate *AP* has to the ordinate *CQ*.

whatever be the number of sides of the regular inscribed polygon which is compared with $V$.

82. COR. An equilateral triangle is the least of all regular figures inscribed in the same circle, and has the least perimeter.

ON

# MAXIMA AND MINIMA.

## PART I.

### SECTION III.

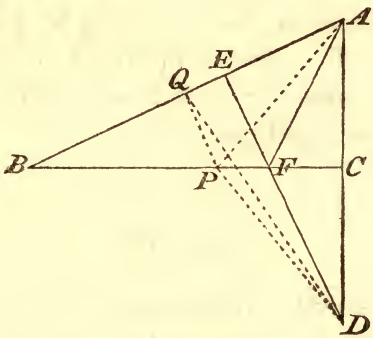
#### Prop. I.

83. *Problem.* In the greater of the two sides, containing the right angle of a scalene right-angled triangle, to find a point from which if two straight lines be drawn, the one to the opposite angle, the other to the hypotenuse, their aggregate shall be a minimum.

Let $ACB$ be a scalene right-angled triangle, right-angled at $C$, and having the side $BC$ greater than $CA$; it is required to find a point in $BC$,

from which if two straight lines be drawn, one to the point *A*, and the other to the hypotenuse *AB*, their aggregate shall be a minimum.

Produce *AC* to *D*, and make *CD* equal to *AC*; from *D* draw (E. 12. 1.) *DE* perpendicular to *AB*,




and let it cut *BC* in *F*; join *F*, *A*; the aggregate of *FE* and *FA* is a minimum.

For, take any other point *P* in *BC*; join *P*, *A*, and draw *PQ* perpendicular to *BA*; then *PQ* will be the shortest (E. 17. and 19. 1.) of all straight lines drawn from *P* to *BA*; also, join *D*, *P*, and *D*, *Q*. Then (E. 4. 1.) *DF* is equal to *FA*, and *DP* to *PA*; and, therefore, *DE* is equal to *EF* and *FA* together; and *DP*, *PQ*, to *AP*, *PQ*; but (E. 17. and 19. 1.) *DQ* is greater than *DE*; and (E. 20. 1.) *DP* and *PQ* are together greater than *DQ*; much more, then, are *DP*, *PQ* greater than *DE*; i. e. *AP* and *PQ* are greater than *AF* and *FE*; and *P* is any point whatever, but *F*,

in $BC$; therefore the aggregate of $AF$, $FE$ is a minimum.

84. Def. The center of the circle inscribed in, or that of the circle described about, any regular polygon, is called the center of that polygon.

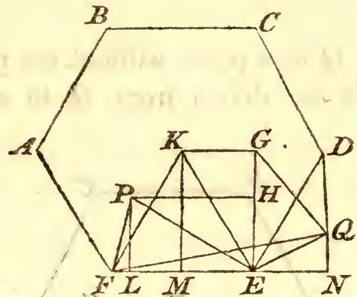
*Remark.* It appears from E. Prop. 13. 4, in which proposition the reasoning is general, that the center of a circle, inscribed in any regular polygon, is also the center of a circle described about the same polygon.

## Prop. II.

85. *Theorem.* If from any point, in a regular polygon, straight lines be drawn perpendicular to the several sides, their aggregate shall be equal to the aggregate of the perpendiculars drawn from the center to the sides: And if perpendiculars be drawn to the sides, from any point without the polygon, their aggregate shall be greater than that of the perpendiculars drawn from the center to the sides.

Let $ABCDEF$ be any regular polygon; find (Art. 35.) its center $K$; and let $P$ be any point within the polygon, and $Q$ any point without it; the aggregate of the perpendiculars drawn from $K$, to the several sides of the figure, is equal to that of the perpendiculars drawn from $P$ to the sides, and is less than that of the perpendiculars drawn from $Q$ to the sides.

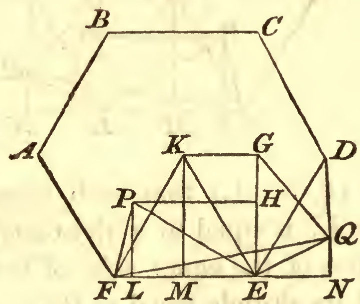
First, the polygon may be divided into as many triangles as it has sides, both by drawing lines from *K*, and from *P*, to each of its angles; and it




is manifest (E. 37. 1.) that each triangle, having its vertex at *K*, is equal to a right-angled triangle which has one of the equal sides of the figure for a base, and its altitude equal to the perpendicular drawn from *K* to that side; and, if these altitudes be placed in the same straight line, it follows from E. 37. 1. that the aggregate of these triangles is equal to a right-angled triangle, having one of the equal sides of the polygon for its base, and the aggregate of the altitudes for its altitude: In the same manner it may be shewn that the aggregate of the triangles, into which the figure is divided by straight lines drawn from *P*, is equal to a right-angled triangle having a side of the polygon for its base, and the aggregate of the perpendiculars drawn from *P* for its altitude; but these two right-angled triangles are equal; for each of them is equal to the whole polygon; also

their bases are equal; wherefore (Art. 19.) their altitudes are equal; i. e. the aggregate of the perpendiculars drawn from $K$ to the sides, is equal to that of the perpendiculars drawn from $P$ to the sides.

But, if $Q$ be a point without the polygon, let straight lines be drawn from $Q$ to each of the

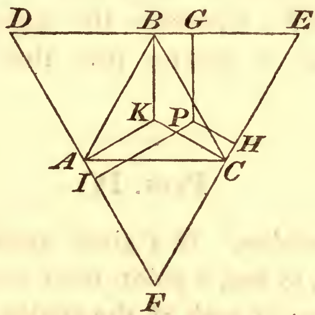



angles of the figure; then, it is manifest that the aggregate of the triangles thus formed, each having its vertex at $Q$ and one of the sides of the figure for its base, is greater than the polygon, and, therefore, greater than the aggregate of the triangles having their summits at $K$; and it may be shewn, by proceeding as in the former case, that, therefore, the aggregate of the perpendiculars drawn from $Q$ to the sides, is greater than that of the perpendiculars drawn from $K$ to the sides.

### Prop. III.

86. *Theorem.* The aggregate of the straight lines, drawn from the center of a regular polygon to each of its angles, is less than that of the straight lines drawn from any other point whatever to each of the angles.

Let *ABC* be any regular polygon of which *K* is the center, and *P* any other point; the aggregate




of the straight lines, drawn from *K* to the angles, is less than that of the straight lines drawn from *P* to the angles.

Join *K, A,* and *K, B,* and *K, C,* &c.; and also *P, A,* and *P, B* and *P, C,* &c.; through *A, B, C,* &c. draw (E. 11. 1.) *FD, DE, EF,* &c. perpendicular to *KA, KB, KC,* &c. respectively. Since *K* is the center of a circle described about *ABC*, the figure* *DEF* is a regular polygon of the same

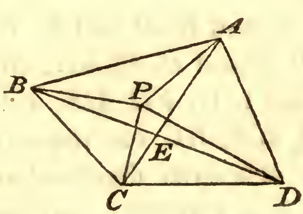
* The construction and proof used by Euclid in Prop. 12. B. 4, is applicable to the description of any regular polygon about a circle, in which a similar polygon has been inscribed.

number of sides as *ABC*, and of which *K* is also the center; from *P* draw (E. 12. 1.) *PG*, *PH*, *PI*, &c. perpendicular to *DE*, *EF*, *FD*, &c. respectively; then (E. 17. 1.) the angles *PBG*, *PCH*, *PAI* are each less than a right angle, and, therefore, (E. 19. 1.) *PA*, *PB*, *PC*, &c. are together greater than *PI*, *PG*, and *PH* taken together; but (Art. 85.) the aggregate of *KA*, *KB*, *KC* is not greater than that of *PI*, *PG*, *PH* whether the point *P* be within, or without, the polygon *DEF*; wherefore the aggregate of *PA*, *PB*, *PC*, &c. is greater than that of *KA*, *KB*, *KC*, &c.

## PROP. IV.

87. *Problem.* In a given quadrilateral rectilineal figure, to find a point, from which if straight lines be drawn to each of the angles of the figure, their aggregate shall be a minimum.

Let *ABC* be the given quadrilateral figure; it




is required to find a point within it, from which if straight lines be drawn to *A*, *B*, *C*, and *D*, their aggregate shall be a minimum.

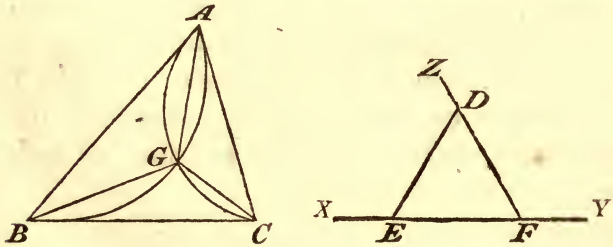
Join *A*, *C* and *B*, *D*; and let *AC* cut *BD* in the point *E*; *EA*, *EB*, *EC*, and *ED* are together less than the aggregate of four straight lines drawn from any other point within the figure to *A*, *B*, *C*, and *D*.

For, let *P* be any other point, and join *P*, *A*, and *P*, *B*, and *P*, *C*, and *P*, *D*. Then (E. 20. 1.) *AP*, and *PC* are together greater than *AC*, and *BP*, and *PD* are together greater than *BD*; wherefore *PA*, *PB*, *PC*, and *PD* are together greater than *AC* and *BD*; i. e. than *EA*, *EB*, *EC*, and *ED*.

## Prop. V.

88. *Problem.* The vertical angle of a triangle being less than the exterior angle of an equilateral triangle, to find within it a point, from which if straight lines be drawn to the three angles of the figure, they shall make equal angles with each other.

Let *ABC* be a triangle having the vertical angle




*A* less than the exterior angle *EDZ* of the equila-

teral triangle *DEF*; it is required to find a point in *ABC*, from which if straight lines be drawn to *A*, *B*, and *C*, they shall make equal angles with each other.

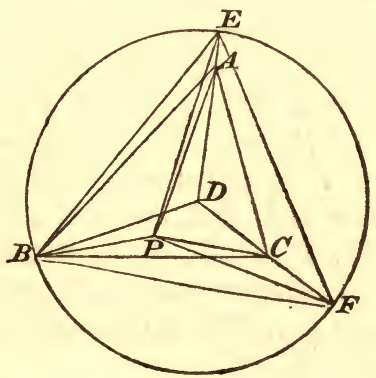
Produce *EF* both ways to *X* and *Y*; upon *AB* and *AC* describe (E. 33. 3.) the circular segments *AGB* and *AGC*, capable of containing each an angle equal to *DEX* or *DFY*, and let them cut each other in the point *G*; join *G*, *A* and *G*, *B*, and *G*, *C*; the angles *AGB*, *BGC*, and *CGA* are equal to each other.

For, since (E. Book 5. 1. Cor.) the angle *DEF* is equal to the angle *DFE*, and that *DEX*, *DEF* are (E. 13. 1.) equal to two right angles, to which also the angles *DFE*, *DFY* are equal, if from these equals be taken the equals *DEF*, and *DFE*, there will remain *DEX* equal to *DFY*; wherefore the angle *AGB* is equal to the angle *AGC*. Again, (E. 32. 1. Cor. 2.) the three angles *DEX*, *DFY*, and *EDZ* are equal to four right angles, to which also (E. 15. 1. Cor. 2.) the angles at *G* are equal; and *AGB*, and *AGC* were made equal to *DEX*, and *DFY*; therefore *BGC* is equal to *EDZ*; and *EDZ* may be shewn to be equal to *DEX*, in the same manner that *DEX* was shewn to be equal to *DFY*; wherefore the angles *AGB*, *BGC*, and *CGA* are equal to each other.

## PROP. VI.

89. *Problem.* * In a given triangle having each of its angles less than the exterior angle of an equilateral triangle, to find a point from which if straight lines be drawn to the three angles, their aggregate shall be a minimum.

Let *ABC* be the given triangle, having each of its angles less than the exterior angle of an




equilateral triangle; it is required to find a point within *ABC*, from which if straight lines be drawn to *A*, *B*, and *C*, their aggregate shall be a minimum.

Find (Art. 88.) the point *D*, at which the straight

* This Problem was, in substance, proposed by FERMAT to TORRICELLI; the method of solution here given is that of VIVIANI.

lines *DA*, *DB*, and *DC* make equal angles with each other; let *P* be any other point in *ABC*, and join *P*, *A* and *P*, *B* and *P*, *C*; then are *DA*, *DB*, and *DC* together less than *PA*, *PB*, and *PC* together.

From the center *D*, at the distance *DB*, the greatest of the straight lines, *DA*, *DB* and *DC*, describe the circle *DEF*, and let it meet *DA* and *DC*, produced, in *E* and *F*; join *B*, *E* and *E*, *F*, and *F*, *B*, and *P*, *E*, and *P*, *F*; then (E. 4. 1.) *EB*, *BF*, and *FE* are equal to each other, and *D* (Art. 84.) is the center of the equilateral triangle *EBF*; wherefore (Art. 86.) *DB*, *DE*, and *DF* are together less than *PB*, *PE*, and *PF* together; but (E. 20. 1.) *PA* and *AE* are, together, greater than *PE*; and *PC* together with *CF* is greater than *PF*; therefore *DB*, *DE* and *DF* are, together, less than *PB*, *PA*, *AE*, *PC* and *CF* together; take away the common parts *AE* and *CF*, and the remainder *DA*, *DB*, *DC* is less than the remainder *PA*, *PB*, *PC*.

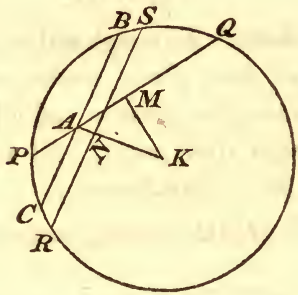
## Prop. VII.

90. *Problem.* Through a given point within a circle, which is not the center, to draw the least chord.

Let *A* be a given point within the circle *PBQC*; it is required to draw through *A* the least chord.

Find (E. 3. 1.) the center *K* of the circle, and

join $A$, $K$; through $A$ draw (E. 11. 1.) the chord




$BAC$ at right angles to $AK$; $BC$ is the least chord which passes through $A$.

For, let $PQ$ be any other chord passing through $A$; from $K$ draw (E. 12. 1.) $KM$ at right angles to $PQ$: then, because $KMA$ is a right angle, $KAM$ (E. 17. 1.) is less than a right angle; wherefore (E. 19. 1.) $KA$ is greater than $MK$, and (E. 15. 3.) $PQ$ is greater than $BC$.

## Prop. VIII.

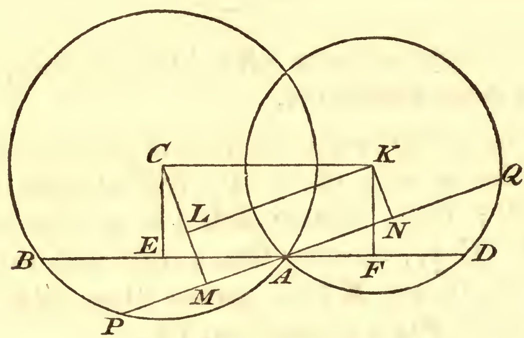
91. *Problem.* To divide a given circular arch into two such parts that the aggregate of their chords shall be a maximum.

It is manifest from the demonstration of Art. 69, that if the given arch be (E. 30. 3.) bisected, the aggregate of the chords of its two parts will be a maximum.

## Prop. IX.

92. *Problem.* Through either of the points of intersection of two given circles, which cut each other, to draw the greatest of all straight lines passing through that point, and terminated both ways by the two circumferences.

Let *APB*, *ADQ* be two given circles, which cut




each other, and let *A* be one of the points of intersection; it is required to draw through *A* the greatest straight line, which is terminated both ways by the circumferences *ABP*, *ADQ*.

Find (E. 1. 3.) the centers *C* and *K* of the two circles, and join *C*, *K*; through *A* draw (E. 31. 1.) the straight line *BAD* parallel to *CK*; *BAD* is the greatest of all straight lines passing through *A*, and terminated both ways by the circumferences of *ABP* and *ADQ*.

For, let *PAQ* be any other straight line so ter-

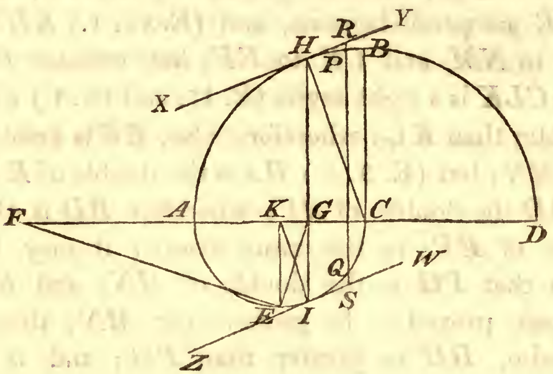
minated, and passing through $A$; from the points $C$, $K$, draw (E. 12. 1.) the straight lines $CE$ and $KF$ perpendicular to $BD$, and $CM$ and $KN$ perpendicular to $PQ$, and through $K$ draw $KL$ parallel to $PQ$; therefore (E. 28. 1.) $LKNM$, and $CKFE$ are parallelograms, and (E. 34. 1.) $KL$ is equal to $NM$, and $CK$ to $EF$; but because the angle $CLK$ is a right angle, (E. 17. and 19. 1.) $CK$ is greater than $KL$; wherefore, also, $EF$ is greater than $MN$; but (E. 3. 3.) $BA$ is the double of $EA$, and $AD$ the double of $AF$; wherefore $BD$ is the double of $EF$; in the same manner it may be shewn that $PQ$ is the double of $MN$; and $EF$ has been proved to be greater than $MN$; therefore, also, $BD$ is greater than $PQ$; and is a maximum.

### Prop. X.

93. *Problem.* If two semi-circles lie on contrary sides of the same straight line, and the radius of the greater be the diameter of the less, to draw the greatest straight line perpendicular to the diameter, and terminated both ways by the two curves.

Let $ABD$, $AEC$ be two semi-circles lying on contrary sides of the straight line $AD$, and let the radius of $ABD$, the greater, be the diameter of $AEC$, the less; it is required to draw the greatest straight line perpendicular to $AC$, and terminated by the curves $AB$, $AEC$.

Find (E. 3. 1.) $K$ the center of the circle $AEC$, and draw (E. 11. 1.) $KE$ at right angles to $AC$; produce $CA$ to $F$, and make $AF$ equal to $AC$; join $F, E$; draw $EG$ perpendicular to $FE$, meet-




ing $AC$ in $G$, and through $G$ draw $HGI$ perpendicular to $AC$; $HGI$ is the greatest of all straight lines which can be drawn perpendicular to $AC$, and which are terminated by $AB$ and $AEC$.

For, let $PQ$ be any other straight line so drawn; draw (E. 17. 3.) the two straight lines $XY$ and $ZW$ touching the circles $ABD$, $AEC$, in the points $H$ and $I$ respectively; produce $PQ$ to meet $XY$ in the point $R$, and $ZW$ in the point $S$; and join $C, H$, and $K, I$; then, (E. 8. 6.)

$$FK : KE :: KE : KG$$

and, *dividendo*,

$$AF : KE :: GC : KG \text{ (E. 17. 5.)}$$

i. e. $HC : KI :: GC : KG$ (by the construct.)

and $HC : GC :: KI : KG$ (E. 16. 5.)

Therefore the two triangles $HGC$, $KGI$, which have the right angles at $G$ equal, have also the sides about two other angles proportionals; and, therefore, (E. 7. 6.) the angle $CHI$ is equal to the angle $HIK$; to $CHI$ add the right angle $CHY$, and to $HIK$ add the right angle $CIZ$, and the whole angle $YHI$ is equal to the whole angle $HIZ$; and they are alternate angles; wherefore (E. 27. 1.) $XY$ is parallel to $ZW$; and (E. 28. 1.) $HI$ is parallel to $RS$; therefore (E. 34. 1.) $HI$ is equal to $RS$; but $RS$ is greater than $PQ$; wherefore, also, $HI$ is greater than $PQ$.

Otherwise:

After having found the center $K$, of the circle $AEC$, trisect (E. 10. 6.) the straight line $KC$, and through $G$, the point of trisection nearest to $K$, draw $HGI$ perpendicular to $AC$; $HGI$ is the maximum required.

For, join $C$, $H$ and $K$, $I$; and since, by the construction, and by the hypothesis, $GC$ is the double of $KG$, and $CH$ is the double of $KI$, therefore,

$$HC : KI :: GC : KG.$$

And the remainder of the demonstration is then the same as in the former method of proof.

94. Cor. 1. Of whatever kind the two curves are, the maximum described in the above proposition may always be found, if two tangents can be drawn to the curves, which are parallel to each other.

95. Cor. 2. Hence, if the two curves be parabolas, and the axis be divided in the ratio of the parameters, the aggregate of the two semi-ordinates, so drawn, will be greatest when it meets the axis in that point of division.

## Prop. XI.

96. *Theorem.* If there be three magnitudes, of which the first is greater than the second, the first shall have to the second, a greater ratio than the first together with the third has to the second together with the third.

Let $A$, $B$, $C$ be three magnitudes, of which $A$ is greater than $B$; $A$ has a greater ratio to $B$ than $A$ together with $C$ has to $B$ together with $C$.

For, $C$ has a greater ratio (E. 8. 5.) to $B$, than it has to $A$: therefore (Note to Art. 36.) $C$ together with $B$ has a greater ratio to $B$, than $C$ together with $A$ has to $A$; and (Note to Art. 78.) $C$ together with $B$ has a greater ratio to $C$ and $A$, together, than $B$ has to $A$; or $A$ has to $B$ a greater ratio than $A$ together with $C$ has to $B$ together with $C$.

97. Cor. If $D$ be a magnitude greater than $C$, $A$ has a greater ratio to $B$ than $A$ and $C$, together, have to $B$ and $D$ together.

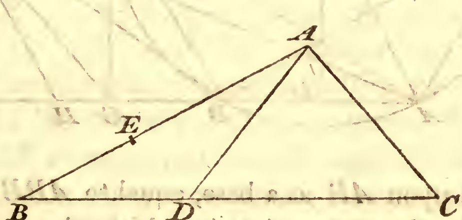
For, the aggregate of $A$ and $C$ has (E. 8. 5.) a greater ratio to that of $B$ and $C$, than to that of $B$ and $D$; much more, then, is the ratio of $A$ to $B$

greater than that of the aggregate of $A$ and $C$ to the aggregate of $B$ and $D$.

### Prop. XII.

98. *Theorem.* If a straight line be drawn from the vertex of a triangle cutting the base, it shall have a greater ratio to either of the segments which it cuts off, than the side adjacent to the other segment has to the base.

Let $ABC$ be a triangle, and $AD$ a straight line



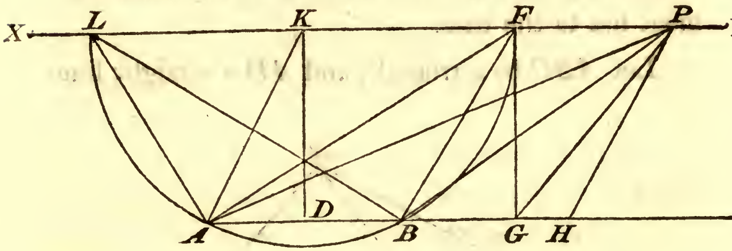

drawn from $A$, cutting the base $AC$ in $D$. $AD$ has a greater ratio to $DC$ than $AB$ has to $BC$.

From $AB$ cut off $AE$ equal to $AD$; then (E. 20. 1.) $AD$ together with $BD$ is greater than $AB$; take $AD$ from $AD$, $BD$, and $AE$, which is equal to $AD$, from $AB$, and the remainder $BD$ is greater than $BE$; wherefore (Art. 96.) $AD$ has a greater ratio to $DC$ than $AB$ has to $BC$.

## Prop. XIII.

99. *Problem.* Of all equal triangles standing upon the same base, to find that in which the ratio of the greater side to the less is a maximum.

Let $APB$ be any triangle standing upon the given base $AB$; of all the triangles which can be




described upon $AB$ as a base, equal to $APB$, it is required to determine that in which the ratio of the greater side to the less is a maximum.

Through $P$ draw (E. 31. 1.) the straight line $XY$ parallel to $AB$, and (E. 39. 1.) it will be the locus of the vertices of all the triangles standing upon the base $AB$, and on the same side of it as $APB$, which are equal to $APB$; bisect $AB$ in $D$ (E. 10. 1.) and through $D$ draw (E. 11. 1.) $DK$ perpendicular to $AB$, and let it meet $XY$ in $K$; join $K$, $A$; and from the center $K$, at the distance $KA$, describe the semi-circle $LAF$, and let it cut $XY$ in $L$ and $F$; join $A$, $L$ and $B$, $L$, and $A$, $F$

and *B, F*; in either of the two equal triangles *ALB, AFB,* the ratio of the greater side to the less exceeds the ratio of *AP* to *PB*.

For draw (E. 12. 1.) *FG* perpendicular to *XY*; then *FG* (E. 16. 3. Cor.) touches the circle *LAF* in *F*, and (E. 29. 1.) cuts *AB* produced at right angles; join *P, G*; *PG* is greater (E. 17. and 19. 1.) than *GF*; also the angle *PAB* is greater than the angle *BPG*; for the two angles cannot be equal; otherwise the two triangles *PGA, PGB* would be similar, and (E. 4. and 16. 6.) the rectangle *AG, GB* would be equal to the square of *GP*; but it also (E. 36. 3.) is equal to the square of the less line *GF*; which is absurd. Neither can the angle *PAB* be less than the angle *BPG*; for then if a straight line be drawn (E. 23. 1.) from *P*, making with *BP* an angle equal to the angle *PAB*; it may in the same manner be shewn, that a less rectangle than *AG, GB,* is equal to a greater square than that of *GF*, which is absurd; wherefore the angle *PAB* is greater than *BPG*, and if *PH* be drawn, making the angle *BPH* equal to *PAB*, the point *H* is beyond the point *G*. The two triangles *AGF, BGF* are similar, as are also the two triangles *AHP, GHP*; wherefore (E. 4. 6.) *AF* : *FB* :: *AG* : *FG*.

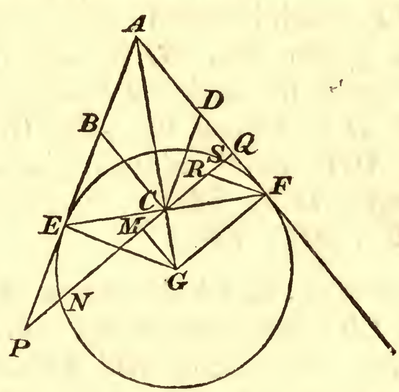
Therefore (E. 22. 6.) the square of *AF* : the square of *FB* :: the square of *AG* : the square of *FG* :: *AG* : *BG*, because *AG, FG*, and *BG* are proportionals; therefore (E. 11. 5.) the square of

*AF* : the square of *FB* :: *AG* : *GB*; and, in the same manner, it may be shewn that the square of *AP* : the square of *PB* :: *AH* : *BH*; but (Art. 96.) the ratio of *AG* to *GB* is greater than the ratio of *AH* to *BH*; wherefore (E. 13. 5.) the square of *AF* has a greater ratio to the square of *FB* than the square of *AP* has to the square of *PB*; but if *Q* be the side of a square, which is a fourth proportional to the squares of *AF*, *FB*, and *AP*, *Q* is less than *PB*; and (E. 22. 6.) *AF* : *FB* :: *AP* : *Q*; wherefore the ratio of *AF* to *FB* is greater (E. 8. 5.) than that of *AP* to *PB*.

### Prop. XIV.

100. *Problem.* Through any of the angular points of a given rhombus, to draw the shortest line terminated by the two sides, produced, which contain the opposite angle.

Let *ABCD* be the given rhombus, and *C* one




of its angular points; it is required to draw

through $C$ the least line, which is terminated by $AB$ and $AD$ produced.

Join $A$, $C$, and through $C$ draw (E. 11. 1.) the straight line $ECF$ perpendicular to $AC$; $ECF$ is the shortest line which can be drawn through $C$, and which is terminated by $AB$, and $AD$ produced.

For, let $PCQ$ be any other straight line drawn through $C$, and terminated, in $P$ and $Q$, by $AB$ and $AD$ produced. Draw $EG$ perpendicular to $AE$, and let it meet $AC$ produced in $G$; join $G$, $F$; and draw $GM$ perpendicular to $PQ$. Then (E. Def. 32. and Prop. 8. 1.) the angle $BAC$ is equal to the angle $DAC$, and the angles $ACE$, $ACF$, are right angles, and $AC$ is common to the triangles $ACE$, $ACF$; therefore (E. 26. 1.) the angle $AEC$ is equal to the angle $AFC$, and $EC$ is equal to $CF$; and $CG$ is common to the two right-angled triangles $ECG$, $FCG$; therefore (E. 4. 1.) $GF$ is equal to $GE$, and the angle $GFE$ to the angle $GEF$; but the angle $AFE$ was before shewn to be equal to the angle $AEF$; wherefore $AFG$ is equal to $AEG$, and is a right angle. From the center $G$, at the distance $GE$, describe the circle $NER$, which will touch $AE$ in $E$, and $AF$ in $F$; let it cut $PQ$ in $N$ and $S$; then, because the angle at $M$ is a right angle, $GC$ (E. 17. and 19. 1.) is greater than $GM$; wherefore (E. 15. 3.) $EF$ is less than $NS$; much more, then, is $EF$ less than $PQ$.

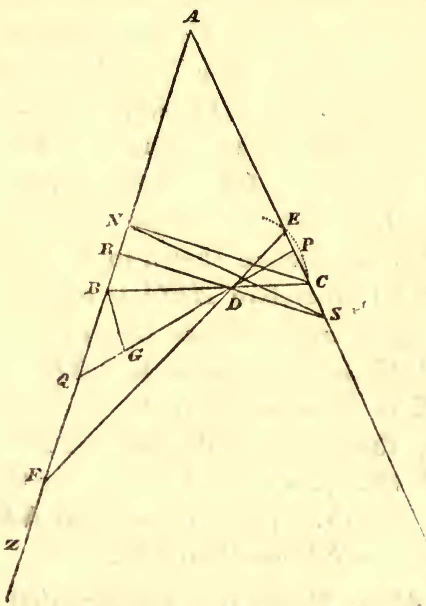
101. Cor. Hence it is manifest that the base of an isosceles triangle is less than any other

straight line, drawn from any point, in either of the two equal sides, through the bisection of the base, so as to meet the other side produced.

## Prop. XV.

102. *Theorem.* If the base of an isosceles triangle be divided into two unequal parts, the base is the least of all straight lines which can be drawn through the point of section, from the side that is nearest to it, so as to meet the other side produced.

Let *ABC* be the given isosceles triangle; let




the base *BC* be divided into two unequal parts in

the point $D$, and let $PQ$ be a straight line drawn through $D$, from any point $P$, in the side $AC$, which is nearest to $D$, meeting $AB$ produced in $Q$: $PQ$ is greater than $BC$.

For, since the angle $APQ$ is greater (E. 16. 1.) than the angle $C$, that is (E. 5. 1. and hypothesis) than the angle $ABC$, therefore (E. 13. 1.) the angle $QBD$ is greater than the angle $DPC$: Make, therefore, (E. 23. 1.) at the point $B$, in $CB$, the angle $CBG$ equal to the angle $DPC$.

And, first, if the angle $A$ be not less than a right angle, the angle $DPC$, and therefore, also, its equal $DBG$, is (E. 16. 1.) an obtuse angle: so that (E. 32. 1. and 19. 1.) $GD$ is greater than $BD$, and $DC$ than $DP$: and, by the hypothesis, $BD$ is greater than $DC$. But (E. 15. 1. and construction) the triangles $GBD$, $DPC$, are equiangular: wherefore (E. 4. 6.)

$$GD : DB :: DC : DP.$$

And $GD$ has been shewn to be greater than $DC$. Wherefore (E. 25. 5.) $GD$ together with $DP$ is greater than $BD$ together with $DC$; that is, $GP$ is greater than $BC$: much more, then, is $PQ$ greater than $BC$.

But, secondly, let the angle $A$ be less than a right angle.

From $D$, as a center, at the distance $DC$ describe a circle, and let it cut $AC$ in $E$; join $E$, $D$; and produce $ED$ to meet $AB$ produced in $F$. Then (E. 17. 1. and hypothesis) since the

angle $ABC$ is an acute angle, the angle $CBF$ is (E. 13. 1.) obtuse: whence (E. 32. 1. and 19. 1.) $DF$ is greater than $DB$; but, by the construction, $ED$ is equal to $DC$: wherefore, the whole $EF$ is greater than the whole $BC$: and, it may, in the same manner be shewn, that a straight line drawn through $D$, from any point between $E$ and $A$, to meet $AB$ produced, is greater than $EF$, and consequently greater than $BC$.

But let, now, $PQ$ be drawn through $D$, from any point $P$, between $E$ and $C$, and let it meet $AB$ produced in $Q$.

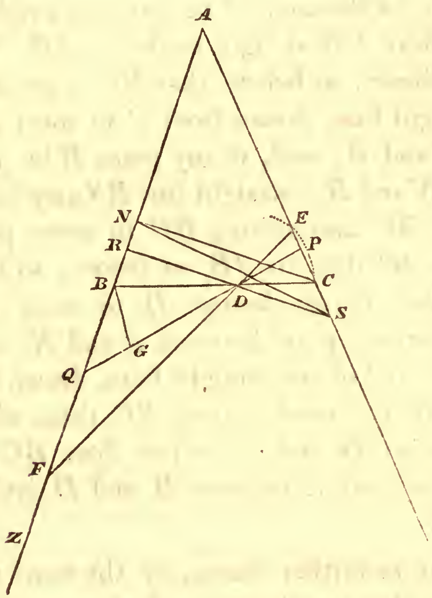
Then, the same construction being made as in the former case, the angle $GBD$ is equal to $DPC$, and the angle $BGD$ to $DCP$; but (E. 16. 1.) the angle $DPC$ is greater than $DEC$, that is (E. 5. 1.) than $DCP$; therefore, the angle $GBD$ is greater than $DGB$, and (E. 19. 1.) the side $GD$ is greater than $BD$. It is manifest, then, that $PQ$ may be shewn to be greater than $BC$, exactly in the same manner, as $PQ$ was shewn to be greater than $BC$ in the former case.

## SCHOLIUM.

If a straight line be drawn through a given point, which divides the base of an isosceles triangle unequally, from the side which is the *further* from that point, to meet the side which is the nearer to it,

produced, the line so drawn is not *necessarily* greater than the base.

For, let $ABC$ be the given isosceles triangle, and $D$ a given point in the base $BC$, nearer to




$AC$ than to $AB$. And, first, if the angle $A$ be not less than a right angle, it is manifest (E. 16. 1. 32. 1. and 19. 1.) that $BC$ is greater than any straight line that can be drawn from $C$ to $AB$: If, therefore, from any point $R$, in $AB$, as a center, at a distance equal to $BC$, a circle be described, it will cut $AC$ produced, in some point $S$: let $R$, $D$ and $S$ be supposed to be in the same straight line, and let $R, S$ be joined. Then, it is manifest, that of all straight lines drawn from $R$ to

meet $AC$ produced, those which cut $BC$ between $D$ and $C$ will be less than $RS$, that is, than $BC$; and that those which cut $BC$ between $B$ and $D$, will be greater than $BC$.

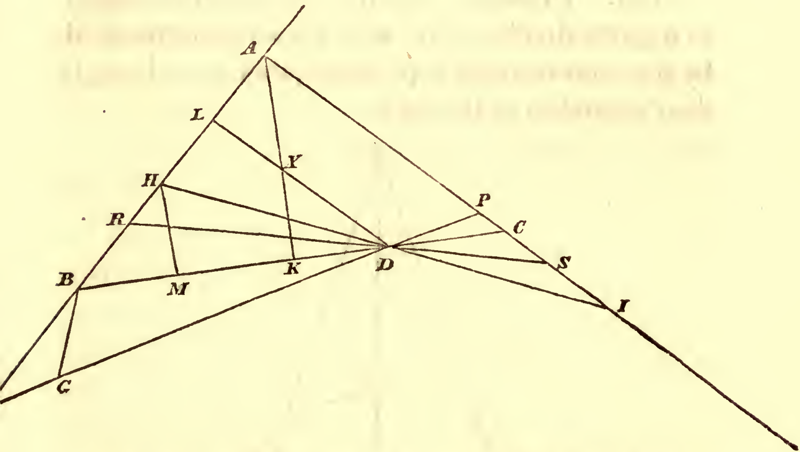
Again, let the angle $A$ be less than a right angle. From $C$ draw $CN$ at right angles to $AB$. Then it may be shewn, as before, that $BC$ is greater than any straight line, drawn from $C$ to meet $AB$ between $N$ and $B$; and, if any point $R$ be assumed between $N$ and $B$, a straight line $RS$ may be drawn, equal to $BC$ and cutting $BC$ in some point $D$, nearer to $AC$ than to $AB$, as before: so that any straight line, drawn through $D$, to meet $AC$ produced, from any point between $A$ and $N$, is greater than $BC$; and of any straight lines, drawn from $R$, to meet $AC$ produced, cutting $BC$, those which cut $BC$ between $D$ and $C$ are less than $BC$; whilst those which cut it between $B$ and $D$ are greater than $BC$.

It may be further shewn, by the same mode of reasoning, that, in all cases, and wherever the point $D$ is assumed, the straight line drawn through $D$ perpendicular to $AB$, and terminated, on the other side, by $AC$ produced, is less than any straight line drawn through $D$, from any point in $AB$, between the intersection of that perpendicular and the vertex $A$, to meet $AC$ produced.

Again, whatever be the species of the isosceles triangle $ABC$, and wherever the point $D$, nearer to $AC$, be assumed, if through $D$, the straight line

*HI* be drawn (Art. 2.) so as to be bisected in *D*, then is *HI* greater than *BC*.

For, draw *HM* and *AK* perpendicular to *BC*; then (E. 5. 1. 26. 1. and hypothesis) *BK* is equal

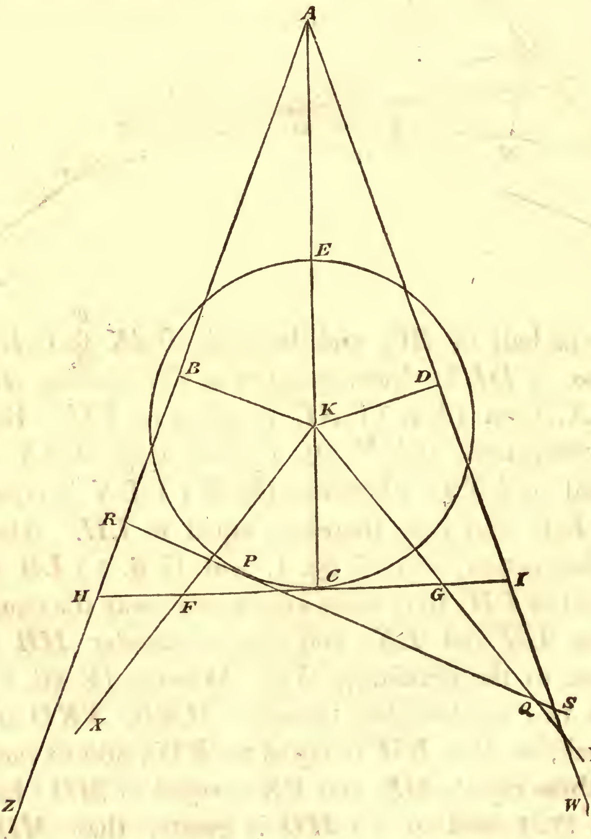



to the half of *BC*, and the angle *BAK* to *CAK*. Also, if *DL* be drawn parallel to *CA*, cutting *AK* in *X*, then (Art. 2.) *AL* is equal to *LH*. But (construction, and E. 29. 1.) the angle *LAX* is equal to *LXA*; wherefore (E. 6. 1.) *LX* is equal to *LA*; and it is, therefore, equal to *LH*. Also, (construction, and E. 29. 1. and E. 6. 1.) *LB* is equal to *LD*: from these equals take away the equal parts *LH* and *LX*; and the remainder *HB* is equal to the remainder *XD*. Whence (E. 26. 1.) the two right-angled triangles *HMB*, *XKD* are equal; so that *BM* is equal to *KD*; add to each of these equals *MK*, and *BK* is equal to *MD*; but (E. 17. 1. and 19. 1.) *HD* is greater than *MD*; therefore *HD* is greater than *BK*, which is equal

to *MD*; and *HI* the double of *HD*, is greater than *BC*, the double of *BK*.

### Prop. XVI.

103. *Problem.* To draw the shortest tangent to a given circular arch, which shall be terminated by the semi-diameters, produced, that pass through the extremities of the arch.




Let *LCM* be a given circular arch, and *KX*,

*KY*, the two produced semi-diameters, which pass through its extremities *L* and *M*: it is required to draw the shortest tangent to *LCM*, which shall be terminated by *KX* and *KY*.

Bisect the arch *LCM* (E. 30. 3.) in the point *C*; and through *C* draw (E. 17. 3.) the tangent *FCG*: *FCG* is the least straight line which can touch the arch *LCM*, and be terminated by *KX* and *KY*.

For, let *PQ* be any other tangent to the arch *LCM*, which is terminated by *KX* and *KY*; and join *K*, *C*: then (E. 18. 3.) the angles *KCF*, *KCG* are right angles; the angle *CKF* is equal (construction, and E. 27. 3.) to *CKG*; and *KC* is common to the two triangles *KCF*, *KCG*; wherefore (E. 26. 1.) the side *KF* is equal to *KG*; so that the triangle *KFG* is isosceles. It is manifest, therefore, from Art. 102, that *PQ* is greater than *FG*: and, in the same manner, it may be shewn, that any other tangent is greater than *FG*.

104. Cor. 1. Hence, if it be required to divide a given circular arch into two parts, so that the aggregate of the tangents of the parts may be a minimum, it is evident that the given arch must be bisected.

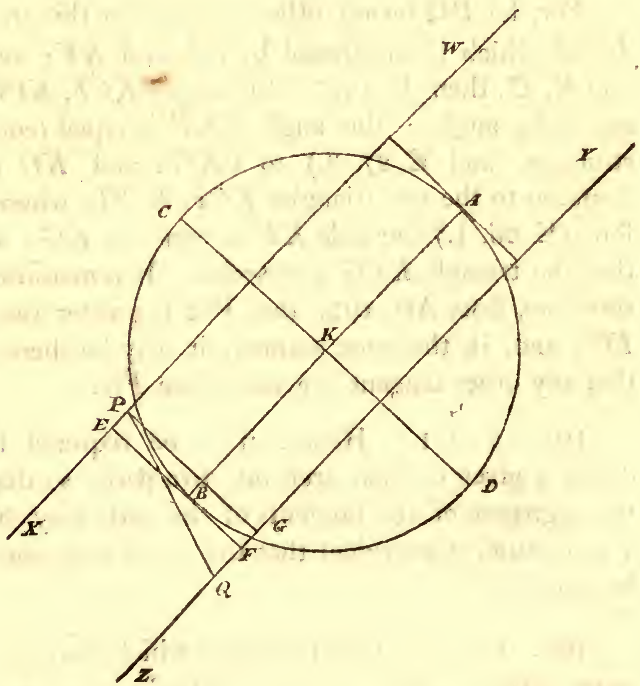
105. Cor. 2. Of all triangles which have the same vertical angle, or equal vertical angles, and the perpendiculars, let fall from the vertex of the

base, also equal, that which is isosceles has the least base.

## Prop. XVII.

106. *Problem.* To draw the shortest tangent to a given circle, which shall be terminated by two given parallel straight lines, that are situated on contrary sides of the center.

Let *ACBD* be the given circle; *K*, its center,




and *WX*, *YZ* two given parallel straight lines, on

contrary sides of the center *K*: it is required to draw the shortest tangent to *ACBD*, that shall be terminated by *WX* and *YZ*.

Through the center *K* draw (E. 31. 1.) the diameter *AKB* parallel to either of the two parallels *WX*, *YZ*; and through *B* draw (E. 17. 3.) the tangent *EBF*: *EBF* is the shortest of all tangents to the given circle, that are terminated by *WX* and *YZ*, and that are situated on the same side of the center *K*.

For, let *PQ* be any other tangent on that side of the center, terminated by *WX* and *YZ*; and through *P* draw *PG* parallel to *EF*; then is *EG* a parallelogram, and (E. 34. 1.) *PG* is equal to *EF*: but (E. 18. 3. and 29. 1.) the angle *PGQ* is a right angle; wherefore (E. 17. 1. and 19. 1.) *PQ* is greater than *PG*, or than *EF*.

And in the same manner may a tangent to the circle, at the other extremity *A*, of the diameter *AB*, which (E. 34. 1.) is equal to *EF*, be shewn to be less than any other tangent on that side of the center.

## Prop. XVIII.

107. *Problem.* To draw the shortest tangent to a given circle, which shall be terminated by two given straight lines that meet one another, and that are equally distant from the center, so as that the tangent and the point of intersection of the given lines, shall be on contrary sides of the center.

Let the two given straight lines $AZ$ and $AW$, which meet in $A$, be equally distant from the center $K$, of the given circle $ELM$. It is required to draw the shortest tangent to the circle, which shall be terminated by $AZ$ and $AW$, and which shall be on the contrary side of $K$, that $A$ is.

Join $A$ and the center $K$*; produce $AK$ to meet the circumference in $C$; and through $C$ draw (E. 17. 3.) the tangent $HCI$: $HCI$ is the least tangent, which was to be drawn.

For, let $RS$ be any other tangent to the circle, on the same side of $K$ that $HI$ is, and terminated, also, by $AZ$ and $AW$. From $K$ draw (E. 12. 1.) $KB$ and $KD$ perpendicular to $AZ$ and $AW$, respectively: then, because (E. Def. 4. B. 3.) $KB$ and $KD$ are equal, and that the angles at $B$ and $D$ are right angles, and that $AK$ is the common hypotenuse of the two right-angled triangles $ABK$, $ADK$, therefore (E. 47. 1.) $AB$ is equal to $AD$, and (E. 8. 1.) the angle $BAK$ is equal to $DAK$: wherefore, since the two right-angled triangles $ACH$, $ACI$, are (E. 18. 3.) right-angled at $C$, the angle $AHC$ is equal (E. 32. 1.) to $AIC$; so that the triangle $AHI$ is isosceles. It is manifest, therefore, from Art. 102, that $RS$ is greater than $HI$: and, in the same manner, may any other tangent on the same side of $K$, be shewn to be greater than $HI$.

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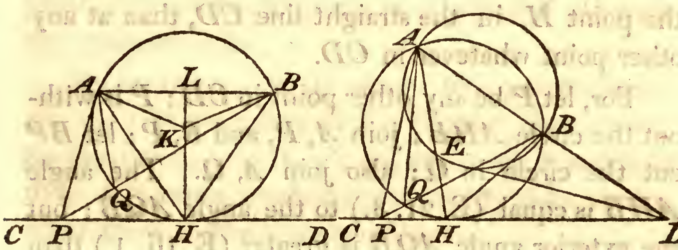
* See the figure in p. 140.

## PROP. XIX.

108. *Problem.* To describe a circle which shall touch a given straight line, and pass through two given points, both on the same side of it, and in the same plane with it.

Let *CD* be a given straight line, and *A, B* two given points without it, both on the same side of *CD*; it is required to draw a circle through *A* and *B*, which shall touch *CD*.

Join *A, B*; and first, let *AB* be parallel to *CD*: Bisect (E. 10. 1.) *AB* in *L*; through *L*




draw (E. 11. 1.) *LH* perpendicular to *AB* or *CD*; join *A, H*; at the point *A*, in *HA*, make (E. 23. 1.) the angle *HAK* equal to the angle *AHK*, and join *KB*; then (E. 6. 1.) *KH* is equal to *KA*, and (E. 4. 1.) *KA* is equal to *KB*; from the center *K*, at the distance *KA*, or *KB*, or *KH*, describe the circle *AHB*; it shall pass through the three points *A*, *H*, and *B*, and (E. 16. 3. Cor.) shall touch *CD* in *H*.

But if *AB* be not parallel to *CD*, let *AB*, pro-

duced, meet *CD* in the point *D*. Upon *AB* as a diameter describe the circle *ABE*, and from *D* draw (E. 17. 3.) the straight line *DE* touching it in *E*; from *DC* cut off *DH* (E. 3. 1.) equal to *DE*, and describe (E. 5. 4.) the circle *AHB* passing through the three points *A*, *H*, and *B*. The circle *AHB*, which passes through *A* and *B*, touches *CD* in *H*.

For (E. 36. 3.) the rectangle contained by *AD* and *DB* is equal to the square of *DE*, and, therefore, is equal also to the square of *DH*, because *DH* was made equal to *DE*; wherefore (E. 37. 3.) the circle *AHB* touches *CD* in *H*.

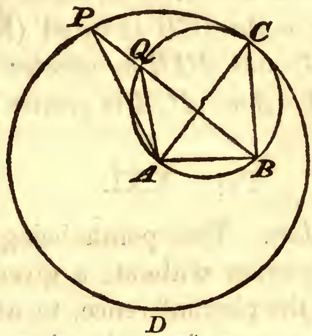
109. Cor. *AB* subtends a greater angle at the point *H*, in the straight line *CD*, than at any other point whatever in *CD*.

For, let *P* be any other point in *CD*; *P* is without the circle *AHB*; join *A*, *P*, and *B*, *P*; let *BP* cut the circle in *Q*; also join *A*, *Q*. The angle *AHB* is equal (E. 21. 3.) to the angle *AQB*; but the exterior angle *AQB* is greater (E. 16. 1.) than the interior opposite angle *APB*; wherefore, also, *AHB* is greater than *APB*.

### Prop. XX.

110. *Problem.* To find a point in the circumference of a given circle, at which any given straight line drawn from the center, but less than a radius of the circle, shall subtend the greatest angle.

Let *CPD* be the given circle, and *AB* a given




straight line, drawn from the center *A*, less than a radius of the circle; it is required to find the point, in the circumference, at which *AB* subtends the greatest angle.

From the point *B* draw *BC* perpendicular (E. 11. 1.) to *AB*, and let it meet the circumference in *C*; join *A*, *C*; the angle *ACB* is greater than any other angle subtended by *AB*, at any other point in the circumference.

For, take any other point *P* in the circumference, and join *P*, *A* and *P*, *B*; and upon *AC* as a diameter, describe the circle *ACB*, * which

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* If two circles meet each other, and the point which is common to both circumferences lie in the same straight line with the centers of both, the circles shall be in contact with each other. This is evident, by drawing from the common point a straight line perpendicular to the line joining the centers, which (E. 16. 3. Cor.) will touch them both; and the converse of E. 31. 3. is readily proved by a *reductio ad absurdum.*

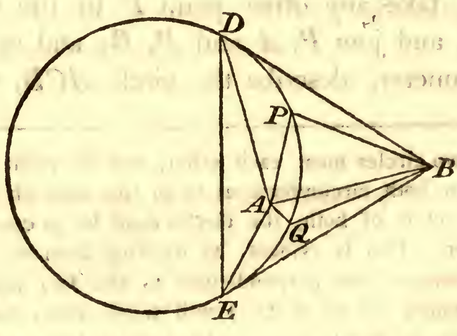
will pass through $B$, and touch the circle $CDP$ in $C$; let $BP$ cut the circle $CAB$ in $Q$; and join $A$, $Q$; then the angle $ACB$ is equal (E. 21. 3.) to the angle $AQB$, and $AQB$ is greater (E. 16. 1.) than $APB$; wherefore $ACB$ is greater than $APB$.

## Prop. XXI.

111. *Problem.* Two points being given, the one within, the other without, a given circle, to find a point in the circumference, to which if two straight lines be drawn from the given points, the one falling upon the concave, the other upon the convex circumference, the angle contained by them shall be a minimum.

Let $DPE$ be the given circle, and $A$, $B$, the two given points; it is required to find a point in the circumference $DPC$ to which, if two straight lines be drawn from $A$ and $B$, the angle contained by them shall be a minimum.

From the point $B$ draw (E. 17. 3.) the two




straight lines $BD$, $BE$, touching the circle in $D$

and $E$; join $A, D$ and $A, E$; and let the angle $DBA$ be *greater than the angle $ABE$; then is the angle $AEB$ less than any other angle, contained by two straight lines drawn from $A$ and $B$, to any other point of the circumference, the one falling upon the concave, the other upon the convex circumference.

For, join $D, E$; and since $BD$ and $BE$ touch the circle $DPE$, they are (E. 36. 3.) equal to each other, and the angle $BDE$ is equal (E. 5. 1.) to the angle $BED$; and because $DB$ is equal to $BE$, and $AB$ is common to the two triangles $DBA$, $EBA$, and that the angle $DBA$ is greater than the angle $EBA$, $DA$ is greater (E. 24. 1.) than $AE$; wherefore the angle $AED$ is greater (E. 18. 1.) than the angle $ADE$; from $BDE$ take $ADE$, and from $BED$, which is equal to $BDE$, take the greater angle $AED$, and the remainder $AEB$ is less than the remainder $ADB$. Again, because $BD$ and $BE$ touch the circle $DPE$, all straight lines drawn from $B$ to the convex circumference must fall within $BD$ and $BE$; in the arch $DPE$, take any point $P$ on the same side of $AB$ with $D$, and any point $Q$ on the same side of $AB$ with $E$, and join $A, P$ and $B, P$, and $A, Q$ and $B, Q$; then (E. 21. 1.) $ADB$ is less than $APB$; and $AEB$ has been shewn to be less than $ADB$; much more,

* The two points $A$ and $B$ are here supposed not to be in the same straight line with the center of the circle; if they be, the two angles $ADB$, $AEB$ will be equal, and either of them will be less than any other angle, according to the specified conditions.

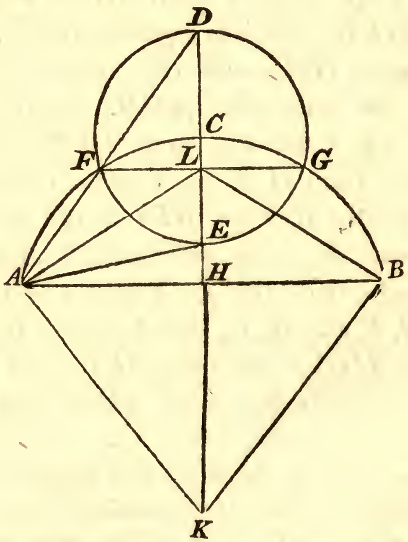
then, is $AEB$ less than $APB$; and (E. 21. 1.) $AEB$ is also less than $AQB$; wherefore $AEB$ is a minimum.

## Prop. XXII.

(112.) *Problem.* To describe a circle which shall touch a given circle, and pass through two given points both without the given circle; the straight line joining the two given points being supposed not to pass through the center of the circle.

Let $DFG$ be the given circle, and $A$, $B$, the two given points without it; it is required to describe a circle which shall pass through $A$ and $B$, and touch the circle $DFG$.

Join $A$, $B$; and find (E. 1. 3.) the center $C$ of



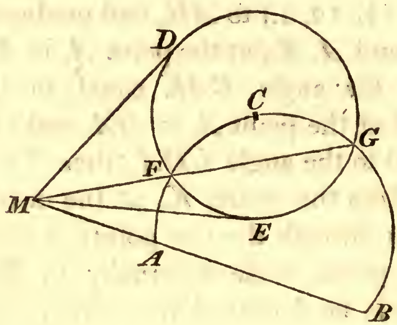

$DFG$; describe (E. 5. 4.) a circle $AFGB$ which

shall pass through the three points $A$, $C$, and $B$; join $F$, $G$; either $FG$ is parallel to $AB$, or it is not; if it be parallel, through $C$ draw $DCH$ perpendicular (E. 12. 1.) to $AB$, and produce it to $K$; join $A$, $D$ and $A$, $E$; at the point $A$, in $EA$, make (E. 23. 1.) the angle $EAK$ equal to the angle $KEA$; and at the point $A$, in $DA$, make the angle $DAL$ equal to the angle $LDA$; then if a circle be described from the center $K$, at the distance $KA$, it shall pass through the two points $A$ and $B$, and touch the given circle externally in $E$; and if another circle be described from the center $L$, at the distance $LA$, it shall pass through $A$ and $B$, and touch the given circle internally in $D$.

For, join $K$, $B$ and $L$, $B$; $AB$ is bisected (E. 3. 3.) in $H$, by $DH$, and the angles at $H$ are right angles; wherefore (E. 4. 1.) $KB$ is equal to $KA$, and $LB$ to $LA$, and (E. 6. 1.) $KE$ is equal to $KA$, and $LD$ to $LA$; therefore the circle described from the center $K$, at the distance $KA$, will pass through the three points $A$, $E$, and $B$; and that described from the center $L$, at the distance $LA$, will pass through $A$, $D$, and $B$; and the circles will be in contact, because the straight line joining their centers passes through the points in which they meet. (Note to Art. 102.)

But if $FG$ be not parallel to $AB$, let them be produced so as to meet in the point $M$; from $M$ draw (E. 17. 1.) the two straight lines $MD$ and $ME$ to touch the given circle $DFG$ in $D$ and

*E*; through *A*, *E*, and *B* describe (E. 5. 4.) a circle, and it shall touch the given circle in *E*;




also describe another circle through the three points *A*, *D*, *B*, and it shall touch the given circle in *D*.

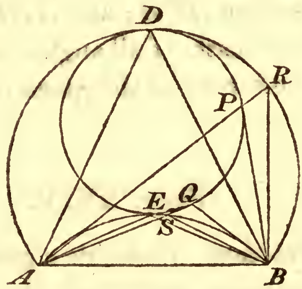
For, (E. 36. Book 3. Cor.) the rectangle *BM*, *MA* is equal to the rectangle *GM*, *MF*; and (E. 36. 3.) the rectangle *GM*, *MF* is equal to the square of *MD*, or *ME*; wherefore, also, the rectangle *BM*, *MA* is equal to the square of *MD*, or *ME*; and (E. 37. 3.) the circle passing through *A*, *B* and *E* is touched by *ME*, which also touches *DEG* in the same point *E*; therefore the circle passing through *A*, *E* and *B* touches the given circle in *E*; and, in the same manner, it may be shewn, that the circle described through *A*, *D* and *B* touches the given circle in *D*.

## Prop. XXIII.

113. *Problem.* *If a given straight line lie wholly without a given circle, and do not when produced, cut the circle, to find the two points in the circumference at which the given line will subtend the greatest and least angles.

Let $DPE$ be the given circle, and $AB$ the given straight line without it; it is required to find the two points in the circumference $DPE$, at which $AB$ shall subtend the greatest and least angles.

Describe (Art. 105,) the circle $AEB$ passing




through $A$ and $B$, and touching $DPE$ externally in $E$, and also the circle $ADB$, touching $DPE$ internally in $D$; $AB$ subtends a greater angle at $E$, and a less angle at $D$, than at any other points in the circumference of $DPE$.

* This is the 34th Proposition of the 6th Book of Pappus; but he has given no construction for describing a circle to pass through two given points, and also to touch another circle.

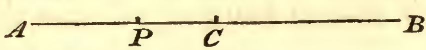
For, join $A, D$ and $B, D$, and $A, E$ and $B, E$; take any other point $P$ such, that $A, P$ and $B, P$, being joined, $AP$ falls upon the concave circumference; and any other point $Q$ such, that $A, Q$ and $Q, B$, being joined, $AQ$ falls upon the convex circumference; produce $AP$ to meet the arch $DB$ in $R$, and let $AQ$ cut the arch $AEB$ in $S$; and join $R, B$, and $Q, B$; the angle $AEB$ is equal (E. 21. 3.) to the angle $ASB$; but (E. 16. 1.) $ASB$ is greater than $AQB$; wherefore $AEB$ is greater than $AQB$; and $AEB$ is also (E. 21. 1.) greater than $APB$.

Again, $ARB$ is less (E. 16. 1.) than $APB$; but (E. 21. 3.) $ADB$ is equal to $ARB$; wherefore $ADB$ is less than $APB$; and $AEB$ is the greatest, and $ADB$ the least, of all angles subtended by $AB$ at the circumference of the given circle $DPE$.

## Prop. XXIV.

114. *Problem.* To divide a given finite straight line into two parts, such that the rectangle, contained by the whole line and one of the parts, shall most exceed the square of that part.

Let $AB$ be the given finite straight line; it is




required to divide it into two such parts that the rectangle, contained by $AB$ and one of the parts, shall most exceed the square of that part.

Bisect $AB$ (E. 10. 1.) in $C$, the excess of the rectangle $AB, BC$ above the square of $BC$ is a maximum.

For, take any other point $P$ in $AB$. The rectangle $AB, BC$ is equal (E. 3. 2.) to the square of $CB$, together with that of $AC$; and the rectangle $AB, BP$ is equal to the square of $PB$, together with the rectangle $AP, PB$; but (Art. 6.) the square of $AC$ is greater than the rectangle $AP, PB$; wherefore the excess of $AB, BC$ above the square of $BC$ is greater than the excess of $AB, BP$ above the square of $PB$.

## Prop. XXV.

115. *Problem.* To divide a given finite straight line into two parts, such that the aggregate of the squares of the two parts may be a minimum.

Let $AB$ (Fig. to Art. 114.) be the given finite straight line; it is required to divide $AB$ into two parts, such that the aggregate of their squares may be a minimum.

Bisect (E. 10. 1.) $AB$ in $C$; the squares of $AC$, and $CB$ are, together, less than the squares of any other two parts, taken together, into which $AB$ can be divided.

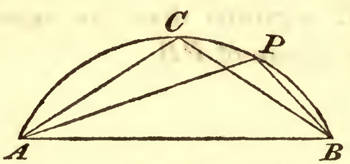
For, take any other point $P$ in $AB$; then (E. 9. 2.) the squares of $AP$, $PB$ are, together, equal to the squares of $AC$, $CB$ and $PC$, together;

wherefore the squares of *AP, PB* are greater than the squares of *AC, CB.*

## Prop. XXVI.

116. *Problem.* To divide a given circular arch into two such parts, that the rectangle contained by their chords shall be a maximum.

Let *ACB* be the given circular arch; it is re-




quired to divide it into two such parts, that the rectangle contained by their chords shall be a maximum.

Bisect (E. 30. 3.) the arch *ACB* in *C,* and join *A, C,* and *B, C*; the rectangle *AC, CB* is a maximum.

For, take any other point *P* in the arch *ACB,* and join *A, P,* and *B, P,* and *A, B.* Then (E. 29. 3.) *AC* is equal to *CB,* and, therefore, *AP* and *PB* are unequal; but (Art. 69.) *AC* and *CB,* i. e. the double of *AC,* are, together, greater than *AP* and *PB*; therefore *AC* is greater than the half of *AP, PB,* and the square of *AC* greater than the square of the half of *AP, PB*; and (Art. 6.) the square of the half of *AP, PB* is greater than

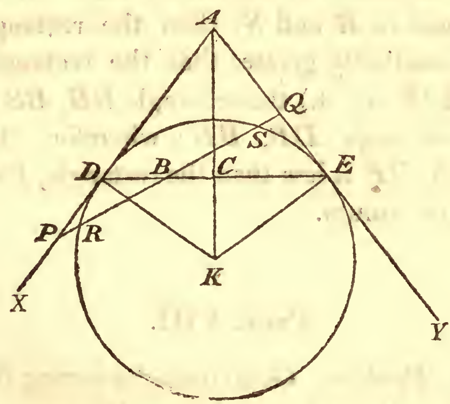
the rectangle $AP$, $PB$; much more, then, is the square of $AC$, i. e. the rectangle $AC$, $CB$, greater than the rectangle $AP$, $PB$.

## Prop. XXVII.

117. *Problem.* Of all triangles having the same vertical angle, and their bases all passing through the same given point, to find that in which the rectangle, contained by the segments into which the given point divides the base, is a minimum.

Let $XAY$ be the vertical angle, and $B$ the given point in the base, common to all the triangles; it is required to find a triangle having $A$ for its vertical angle, and its base passing through the point $B$, and divided by it into two segments, such that the rectangle contained by them is a minimum.

Bisect (E. 9. 1.) the angle $XAY$ by the straight




line $AC$, and from $B$, draw (E. 12. 1.) $BC$ at right

angles to *AC*, and produce it both ways to meet *AX* and *XY*, in *D* and *E*; the rectangle *DB*, *BE* is a minimum.

For, draw (E. 11. 1.) *DK* perpendicular to *AD*, and join *KE*: Because, by the construction, the angles at *C* are right angles, and the angles *DAC*, *EAC* are equal, and *AC* common to the two triangles *ACD*, *ACE*, *AD* (E. 26. 1.) is equal to *AE*: Again, because *AD* is equal to *AE*, and *AK* common to the two triangles *ADK*, *AEK*, and the angle *DAK* is equal to the angle *EAK*, *DK* is equal (E. 4. 1.) to *KE*, and the angle *AEK* to the right angle *ADK*: From the center *K*, at the distance *KD*, describe the circle *DER*; and it will, therefore, pass through *E*, and (E. 16. 3. Cor.) touch *AX* and *XY*, in *D* and *E*.

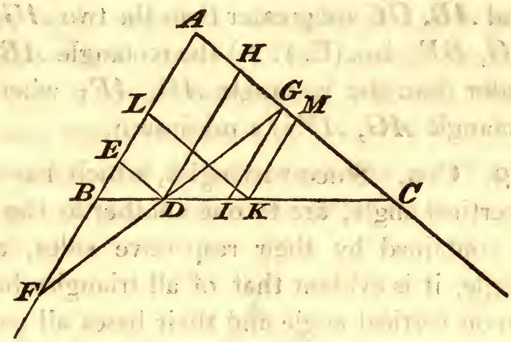
Let now *PQ*, passing through *B*, be the base of any other triangle, having *A* for its vertical angle; *PQ* must cut the circle; let it cut the circumference in *R* and *S*; then the rectangle *PB*, *BQ* is manifestly greater than the rectangle *RB*, *BS*; and (E. 35. 3.) the rectangle *RB*, *BS* is equal to the rectangle *DB*, *BE*; wherefore the rectangle *DB*, *BE* is less than the rectangle, *PB*, *BQ*; and is a minimum.

### Prop. VIII.

118. *Problem.* Of all triangles having the same vertical angle, and their bases all passing through

the same given point, to find that in which the rectangle contained by the sides is a minimum.

Let *FAC* be the given vertical angle, and *D* the given point in the base, common to all the




triangles; of all triangles having the same vertical angle *A*, and having their bases drawn through *D*, it is required to find that in which the rectangle contained by the sides is a minimum.

Through *D* draw (Art. 1.) the straight line *FDG*, which is bisected in *D*, and terminated by *AF* and *AC*; the rectangle *AG*, *AF* is a minimum.

For, draw through *D* any other straight line *BDC*, terminated by *AF* and *AC* in *B* and *C*; also draw (E. 31. 1.) *GI* parallel to *AB*; then it may be shewn, as in Art. 3. that *GI* is equal to *BF*; and, (E. 4. 6.) because the two triangles *CAB*, *CGI* are equiangular, *CA* : *AB* :: *GC* : *GI* or *BF*; therefore (E. 16. 6.) the rectangle *AB*, *CG*

is equal to the rectangle *AC, BF*; but the rectangle *AC, BF* is greater than the rectangle *AG, BF*; wherefore the rectangle *AB, CG* is greater, also, than the rectangle *AG, BF*; add to both the rectangle *AB, AG,* and the two rectangles *BA, AG* and *AB, GC* are greater than the two *AG, BA* and *AG, BF*; i.e. (E. 1. 2.) the rectangle *AB, AC* is greater than the rectangle *AG, AF*; wherefore the rectangle *AG, AF* is a minimum.

119. COR. Since * triangles, which have the same vertical angle, are to one another as the rectangles contained by their respective sides, about that angle, it is evident that of all triangles having a common vertical angle and their bases all passing through the same given point, that is the least which has its base bisected in the given point.

## PROP. XXIX.

120. *Problem.* To find the greatest of all triangles having the same given vertical angle, and

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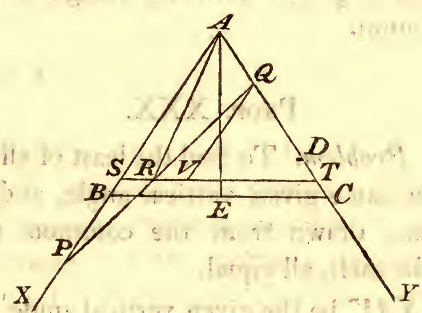
* It is easily proved, by the help of E. 1. 6. that if there be four lines which are proportionals, and four other lines which are also proportionals, the rectangle contained by the first and fifth has to the rectangle contained by the second and sixth, the same ratio, which the rectangle contained by the third and seventh, has to the rectangle contained by the fourth and the eighth: And, thence, it may be deduced as a corollary from E. 23. 6. and 41. 1. that triangles, which have equal vertical angles, are to one another as the rectangles contained by the sides about those equal angles.

the direct distances, between the vertex and the bisection of the base in each, all equal.

Let $XAY$ be the given vertical angle, and $AD$ equal to the direct distance between the vertex and the bisection of the base in each of the triangles; it is required to determine the greatest triangle which has $A$ for its vertical angle, and the direct distance between $A$ and the bisection of its base equal to $AD$.

Draw (E. 9. 1.) the straight line $AE$ bisecting the angle $XAY$, and make $AE$ equal to $AD$;




through the point $E$ draw (E. 11. 1.) the straight line $BEC$ perpendicular to $AE$, meeting $AX$ and $AY$ in $B$ and $C$; the triangle $ABC$ is a maximum.

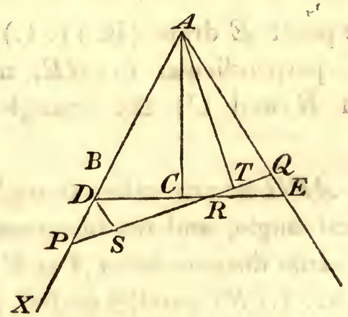
For, let $APQ$ be any other triangle having $A$ for its vertical angle, and the bisection, $R$, of its base, at the same distance from $A$ as $E$ is; through $R$ draw (E. 31. 1.) $ST$ parallel to $BC$, and through $Q$ draw $VQ$ parallel to $AP$; then, because $QV$ is

parallel to $AP$, the angle $VQP$ is equal (E. 29. 1.) to the angle $QPS$; and (E. 15. 1.) the two vertical angles $QRV$, $SRP$ are equal, and the side $QR$ is equal to the side $RP$; wherefore (E. 26. and 4. 9.) the triangle $QRV$ is equal to the triangle $PRS$; add to both the trapezium $AQRS$, and the trapezium $AQVS$ is equal to the triangle $APQ$; but the triangle $ABC$ is greater than the trapezium $AQVS$; for the point $R$ must fall above $BC$, because it is in the circumference of a circle described from the center $A$, at the distance $AD$, which touches $BC$ in $E$; wherefore, also, the triangle $ABC$ is greater than the triangle $APQ$, and is a maximum.

## Prop. XXX.

121. *Problem.* To find the least of all triangles having the same given vertical angle, and the perpendiculars, drawn from the common vertex to the base in each, all equal.

Let $XAY$ be the given vertical angle, and $AB$




equal to the perpendicular drawn from the vertex

to the base in any one of the triangles; it is required to find the least triangle which can have $A$ for its vertical angle, and the perpendicular drawn from $A$ to its base equal to $AB$.

Bisect (E. 9. 1.) the angle $XAY$ by the straight line $AC$, and make $CA$ equal to $AB$; through $C$ draw (E. 11. 1.) the straight line $DCE$ perpendicular to $AC$, and let it meet $AX$ and $AY$ in $D$ and $E$; the triangle $ADE$ is a minimum.

For, let $APQ$ be any other triangle having $A$ for its vertical angle, and the perpendicular $AT$ drawn from $A$ to its base $PQ$, equal to $AC$ or $AB$. It is manifest from E. 17. 1. that $PQ$ cannot pass through $C$; let it cut $DE$ in $R$; it is further evident from the construction, and E. 27. 1. that $DC$ is equal to $CE$; also (E. 16. 1.) the angle $PDE$ is greater than the angle $DAE$; if, therefore, through $D$, $DS$ be drawn parallel to $AE$, it will meet $PQ$ between $P$ and $R$; and (E. 15. and 29. 1.) the two triangles $DRS$, $QRE$ are similar; therefore (E. 19. 6.) the triangle $DRS$ is to the triangle $QRE$ as the square of $DR$ to the square of $RE$; but $DR$ is greater than $RE$, and the square of $DR$ greater than the square of $RE$; wherefore the triangle $DRS$ is greater than the triangle $QRE$; but the triangle $PRD$ is greater than $DRS$; much more, then, is $PRD$ greater than $QRE$; add to both the trapezium $AQRD$, and the triangle $APQ$ is shewn to be greater than the triangle $ADE$; therefore the triangle $ADE$ is less than any other such triangle; i. e. it is a minimum.

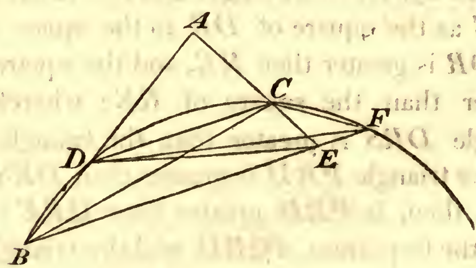
Otherwise:

Let $XAY$ be the given vertical angle, and $AB$ equal to the perpendicular drawn from $A$ to the base of any one of the triangles: Then, it is manifest, from the hypothesis, that if from $A$ a center, at the distance $AB$, a circle be described, the bases of the triangles will be tangents to it: And, if $DE$ be drawn touching the arch contained between $AX$ and $AY$ in it's bisection $C$, $DE$ (Art. 105.) is the least base: Therefore, since the triangles have equal altitudes, the triangle $ADE$ is (E. 1. 6. Cor.) the least triangle.

## Prop. XXXI.

122. *Problem.* Of all equal triangles, having their vertical angles also equal, that which is isosceles has the least perimeter.

Let $ABC$ be any one of a set of equal triangles,




each having its vertical angle equal to $BAC$, and let the triangle $ABC$ be scalene; its perimeter is

greater than that of an equal isosceles triangle having its vertical angle equal to *BAC*.

Find (E. 13. 6.) *AD* a mean proportional between *AB* and *AC*, and produce *AC* to *E*, so that *AE* is equal to *AD*; join *D*, *E*, *D*, *C* and *B*, *E*; and about the triangle *DBC* describe the circle *BDCF*; its circumference will pass beyond the point *E*; for it cannot pass through *E*, otherwise the angle *DBC* would (E. 21. 3.) be equal to the angle *DEC*; but *DEC* is equal (E. 5. 1.) to the angle *ADE*, and, therefore, (E. 16. 1.) greater than *DBC*; neither can the point *E* be beyond the circumference; for then the exterior angle of a triangle would be less than the interior opposite angle, which (E. 16. 1.) is impossible; produce, therefore, *BE* to meet the circumference in *F*, and join *D*, *F* and *C*, *F*; and because, by the construction, $AB : AD :: AE : AC$, therefore (E. 17. 5.) $AB : BD :: AE : CE$, and (E. 2. 6.) *DC* is parallel to *BE*; therefore (E. 29. 1.) the angle *DCB* is equal to the angle *CBF*, and (E. 26. and 29. 3.) *DB* is equal to *CF*; also the angle *DBC* is equal (E. 21. 3.) to the angle *DFC*, and the angle *DCB* (E. 27. 3.) equal to the angle *CDF*, and *DC* is common to the two triangles *DBC*, *DFC*; wherefore (E. 26. 1.) *BC* is equal to *DF*; and *DB* was shewn to be equal to *CF*; therefore *DB*, *BC* are, together, equal to *DF*, *FC*, together; but (Art. 13.) *DF*, *FC* are, together, greater than *DE*, *EC*; wherefore *DB*, *BC* are also greater than *DE*, *EC*; add to both *AD*, and *AC*;

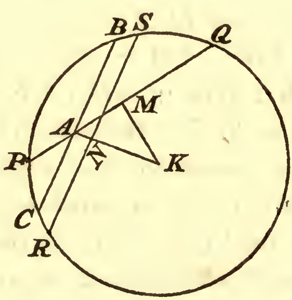
then *AB, BC, AC* are, together, greater than *AD, DE, EA*; i. e. the perimeter of the scalene triangle *ABC* is greater than that of the equal isosceles triangle *ADE*, having the same vertical angle.

123. Cor. An isosceles triangle is greater than any other triangle of equal perimeter, and having also an equal vertical angle.

## Prop. XXXII.

124. *Problem.* Through a given point in a given circle, to draw a chord which shall cut off the least segment.

Let *PBQ* be the given circle, and *A* a given




point within it; it is required to draw through *A* a chord which shall cut off the least segment.

Find (E. 1. 3.) the center *K* of the circle; join *A, K*; and through *A* draw (E. 11. 1.) the chord *CAB* perpendicular to *AK*; the segment *CPB* is a minimum.

For, through $A$ draw any other chord $PAQ$; and from the center $K$ draw (E. 12. 1.) $KM$ perpendicular to $PQ$; and because $AMK$ is a right angle, $AK$ is greater (E. 17. and 19. 1.) than $KM$; from $KA$ cut off (E. 3. 1.) $KN$ equal to $KM$; and through $N$ draw (E. 1. 1.) the chord $RNS$ at right angles to $NK$; therefore (E. 14. 3.) $RS$ is equal to $PQ$; and (E. 28. 26. and 24. 3.) the segment $RPS$ is equal to the segment $PBQ$; but the segment $CPB$ is less than the segment $RPS$; wherefore, also, the segment $CPB$ is less than the segment $PBQ$; i. e. the segment cut off by $CB$ is less than the segment cut off by any other chord, $PQ$, passing through the point $A$.

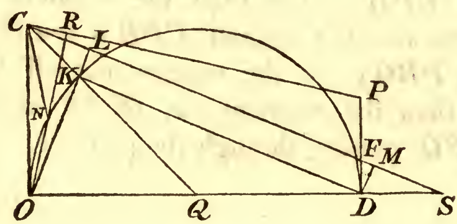
## SCHOLIUM.

Many more propositions, of the same kind, might have been added, if those already given in this Section had not been thought sufficient, both in number and variety, to recommend the subject to the attention of the Student in Geometry. In the several branches of Natural Philosophy a great number of problems, relating to Maxima and Minima, will offer themselves to his invention; and he will do well to exercise his ingenuity in discovering geometrical solutions to them. They will seldom be either more difficult or more tedious, when treated in this manner, than when they are solved by fluxional calculation.

The following questions may serve to exemplify what is here recommended.

(I.) Of all truncated right cones, of the same base and of the same altitude, and moving in a fluid in the direction of a common axis, to find that which meets with the least resistance.

Let *OC* be the semi-diameter of the given base, and *OD* the common altitude, of all the truncated




right cones. * Bisect *OD* in *Q*, and join *CQ*; produce *QD* to *S*, and make *QS* equal to *QC*; join *C*, *S*; from *D* draw *DF* perpendicular to the common axis *OS*; the resistance on the truncated right cone generated by the revolution of the trapezium *FDOC* is a minimum.

For, draw *DM* perpendicular to *CS*; and from the center *Q*, at the distance *QO*, or *QD*, describe the circle *OKD* cutting *CQ* in *K*; join *O*, *K* and *K*, *D*, and produce *OK* to meet *CS* in *L*;

* This is the construction given, without an investigation, by NEWTON, in the Scholium to the 34th Proposition of the Second Book of the *Principia*.

then (E. 5. 1.) the angle *QKD* is equal to the angle *QDK*, and the angle *QCS* to the angle *QSC*; wherefore (E. 32. 1.) the exterior angle *CQO* is the double, both of *QKD* and *QCS*; therefore the angle *QKD* is equal to the angle *QCS*, and (E. 28. 1.) *KD* is parallel to *CS*; but (E. 31. 3.) *OKD* is a right angle; wherefore *OLS* (E. 29. 1.) is also a right angle.

By the common principles of the motion of a body in a resisting medium, the resistance on the surface of the truncated cone is to the resistance on its base as the aggregate of the squares of *DM* and *CL* is to the square of *CO*; i. e. as the aggregate of the squares of *KL* and *LC*, (E. 34. 1.) or of the square of *CK* (E. 47. 1.), to the square of *CO*; and if *CPDO* be the trapezium by the revolution of which any other truncated right cone of the same given base and altitude is generated, and *OR* be drawn at right angles to *CP*, cutting the circle *OKD* in *N*, and *C, N* be joined, the resistance on this truncated cone is to the resistance on its base, as the square of *CN* is to the square of *CO*; but (E. 8. 3.) *CK* is less than *CN*, and the square of *CK* is, therefore, less than the square of *CN*; wherefore (E. 10. 5.) the resistance on the surface of the truncated cone generated by the revolution of *CFDO*, is less than that on the surface of any other such cone.

(II.) A given weight being appended to a point in the axis of a cylindrical lever, at a given dis-

tance from one of its extremities which is the center of motion, and the diameter and specific gravity of the lever being also given, to determine the length of the lever, so that the force applied at its other extremity, to raise the given weight, may be a minimum.

Let the given distance from the center of motion at which the weight is hung be called $2a$, and let $2x$ be assumed the length of the lever; call the force applied at the extremity of the lever, which is to be a minimum, $P$; let the weight of one inch of the lever be unity, and let the given weight be $(A)$ such unities; therefore the weight of the whole lever will be $2x$; and from the Principles of Mechanics,

$$P \times 2x = 2x \cdot x + A \cdot 2a\,; \;\therefore P \cdot x = x^2 + A \cdot a.$$

Let $y$ be a fourth proportional to $x$, $a$ and $A$, so that (E. 16. 6.) $A \cdot a = xy$; $\therefore P.x = x^2 + xy$;

$$\therefore P = x + y.$$

But the rectangle contained by $x$ and $y$ is a given quantity; wherefore (Art. 26.) the sum of $x$ and $y$ is a minimum when $x = y$; that is, $P$ is a minimum when $x$ is a mean proportional between $A$ and $a$.

(III.) In the same manner, if it be inquired what curve, by its revolution, generates a surface best adapted for a Speaking, or a Hearing Trumpet, the question will be found to depend upon the same geometrical proposition as the preceding;

a quantity of the form $x + \frac{A \cdot a}{x}$ must be a minimum; that is, $x$ must be a mean proportional between $A$ and $a$; $A$, $x$, and $a$ being three contiguous portions of the air included in the tube, which is an elastic fluid; whence it is manifest that the curve sought is the Logarithmic curve.

(IV.) Lastly, An horizontal space being given, to find the perpendicular altitude through which a heavy body must fall from rest, so that afterwards describing the horizontal space, with its acquired velocity, the whole time of its motion shall be a minimum.

Let $(a)$ denote the horizontal space, $(m)\overset{\text{feet}}{16\tfrac{1}{12}}$, and $(x)$ the required perpendicular altitude; then, from the Principles of Mechanics, the time of the body's motion is denoted by $\frac{\sqrt{x}}{\sqrt{m}} + \frac{1}{2}\frac{a}{\sqrt{mx}}$; and this quantity is to be a minimum; therefore, also, its square $\frac{x}{m} + \frac{a}{m} + \frac{a^2}{4mx}$ must be a minimum, which it will be when $x + \frac{a^2}{4x}$ is least; that is, as before, when $x = \frac{a}{2}$.

This proposition is best demonstrated synthetically, by the help of the parabola. It is, however,

not unworthy of remark, that the solution of this, and of a great variety of problems of the same kind, may be reduced to the simple geometrical theorem, which asserts the perimeter of a square to be less than the perimeter of any other equal rectangle.

ON

# MAXIMA AND MINIMA.

## PART II.

### ON THE ALGEBRAICAL INVESTIGATION OF MAXIMA AND MINIMA.

## SECTION I.

### ON THE BINOMIAL THEOREM.

THE Binomial Theorem forms the basis of the algebraical investigation of Maxima and Minima. Unless, indeed, it be first legitimately established, neither the Principles of Fluxions, nor those of any other equivalent system, which furnishes the same rules of computation, can be freed from very weighty objections. It is, therefore, most intimately connected with this part of our subject. If, however, what has already been written and

published concerning this theorem, might be considered as satisfactory, it need not have occupied a place in this Treatise. But the truth is, that the commonly received demonstrations of it have no just pretensions to logical exactness. It is, indeed, surprising that men of great talents and attainments have either overlooked the defects of the proofs which they have given of this theorem, or else have preferred the publication of an imperfect chain of reasoning, to the confession of their inability to be rigorously exact. BARON MASERES alone seems to have been scrupulously anxious to attain absolute precision on this point; and, where his laborious endeavours fail of success, he is the first to acknowledge the failure. These assertions can be justified only by a brief review of the methods of investigating the Binomial Theorem, which are commonly thought to be the best.

The proof, of which LANDEN was the author, seems to have been for a long time in great esteem with the algebraists of this country. It was exhibited, under its most advantageous form, in the Philosophical Transactions of the year 1796; from which it has been copied, and re-copied, in several publications that have appeared since.

The principal objections under which it labours are these.

1. There is no previous definition laid down of what is meant by the expansions of $(a \pm x)^{\frac{m}{n}}$, $(a \pm x)^{-m}$, and $(a \pm x)^{-\frac{m}{n}}$, nor is any enquiry made

into the general nature of those expansions, to justify the assumption of the series with which the proof sets out.

2. The proof proceeds upon the supposition of a numerical equality between $(a \pm x)^{\frac{m}{n}}$, $(a \pm x)^{-m}$, $(a \pm x)^{-\frac{m}{n}}$, and their respective expansions continued without limit, which equality does not exist. There is not, necessarily, even an approximation in value between those quantities and their several expansions, when particular numbers are put in the places of $a$ and $x$.

3. The sophism of *shifting the hypothesis* is next introduced; that is, results are obtained by making a supposition in one part of the demonstration, which, if it had been made in a preceding part, would have wholly stopt the process. The value of $x$ is made equal to that of $y$ in one equation, after building upon another equation, in which if that supposition had been made, the latter equation could not have been obtained.

4. The equality of the coefficients of the corresponding powers of $x$ in two series of the form $A + Bx + Cx^2 + \&c. = a + bx + cx^2 + \&c.$ is asserted; and as this is asserted without proof, it is to be supposed that the common method of inferring that equality, by making $x = 0$, which proceeds upon a *petitio principii*, is deemed sufficient.

One, or more of these principal objections may

be made to most of the other proofs of the Binomial Theorem which are best known.

That published by LAGRANGE in his *Theorie des Fonctions Analytiques*, has been rendered much less objectionable by the learned author of the *Principles of Analytical Calculation*, in whose work it appears. Still, it hardly seems judicious to employ the symbol of equality, where no equality exists, although the reader is forewarned of it. This is, in reality, not merely an extension, but a change, of the meaning of the sign. There is, perhaps, some impropriety in denoting by it the relation subsisting between a variable quantity and its limit; still less, then, ought to be used where no such approximation necessarily takes place. There are degrees of inequality, but none of equality. An objection of such a nature, however, is comparatively light. But the proof is made to depend on the assumption, that if $(a + x + z)^{\pm\frac{m}{n}}$ be expanded as a binomial, first by considering $a + x$ as one quantity, then by taking $x + z$ as one quantity, the resulting series shall be *identical*, whatever the index of the expansion be, whether integral or fractional, positive or negative. That this is true when the index is a positive integer, will be readily granted; for it may be intuitively perceived. But can it fairly be assumed to obtain in any other case? How is it that the mind assents to any general proposition? It instantaneously verifies the included assertion, or

negation, by having recourse to definitions, or to some obvious instances to which the proposition is applicable, and in which it is manifestly true, independently of any particularity belonging to any one of the instances considered. If there be not this perception of the agreement, or disagreement, of ideas, no genuine conviction can follow. The plausibility of the énunciation of a proposition may, indeed, be such as to win a hasty consent from the indolent; but, without that necessary verification, there can be no real knowledge. That "the same result must be had, when the same algebraical operation has been performed on the same quantity" is a proposition, at first sight, sufficiently plausible; but in the application made by LAGRANGE of that assertion, the quantity operated upon cannot, strictly speaking, be said to be the same quantity in all the cases; and when the proposition is more precisely enunciated, it becomes necessary to resort to the usual method, in order to judge whether it be true or false. Now $\pm \frac{m}{n}$ is the index of several operations, and there is none of them which *manifestly* produces two identical series, when $(a+x)+z$ and $a+(x+z)$ have been subjected to it. In so simple a case as that in which the index is $-1$, the resulting series are far from being manifestly the same; and they are in no case necessarily so, excepting that in which the index is a whole positive number; for then only is there a numerical equality between

$(a+x+z)^m$ and its whole expansion. The assertion is, undoubtedly, true; but it requires, and admits of, a proof, as much as the Binomial Theorem itself; and it does not appear that any advantage would be gained by *previously* establishing the truth of this assertion.

The faults, here imputed to LANDEN and LAGRANGE, seem to have arisen from the desire of being concise, where conciseness is not attainable, and where precision should have been chiefly aimed at; and from the affectation of generalizing too hastily, where all the included particulars do not readily occur to the mind, and where they have scarcely enough in common, to furnish the basis of a demonstration equally applicable to each of them.

A proof very lately published, in the second part of the Philosophical Transactions, for the year 1816, which sets out from the assumption, that $(a+x)^m \cdot (a+y)^m = [(a+x) \cdot (a+y)]^m$, whether $m$ be positive or negative, integral or fractional, is evidently liable to some of the objections which have been urged against the methods of LANDEN and LAGRANGE. The equation thus assumed for the foundation of a general proof, is intended to comprehend some cases, in which, strictly speaking, no equality obtains. Or, if it be otherwise explained, it supposes, in the form of the results of the operations, designated by the index $m$, an identity which is far from being self-evident.

Considered with respect to exactness, Professor ROBERTSON's proof of the Binomial Theorem, has great merit; and this is the place to acknowledge that his manner of inferring the form of the coefficients of the expanded binomial, when the index is a positive fraction, has been adopted in the following attempt. The author had, indeed, previously applied the same principle, in the case of a negative index; but, before he had perceived its application to the former case, it fell in his way to see the Professor's demonstration. It is remarkable that Mr. Robertson himself is not exact, when he investigates the expansion of $(a \pm x)^{-m}$. He does not obtain it without the same shifting of the hypothesis, the same reduction of the value of $x$ to 0, a supposition implicitly excluded in the previous reasoning, which has been noted in Landen's proof. He also places signs of equality between quantities which are not equal.

After what has been said, the reader will not look for conciseness in the following demonstration; but the Author has studiously endeavoured to avoid the imperfections which have here been pointed out. His original intention was merely to supply the defects, and to rectify the errors of other writers better known than himself. It appeared to him to be necessary, both to the correction of Landen's proof, and to the demonstration of the assumption made by Lagrange, that the ratio of any two successive coefficients, in

every case of the expansion of $(a \pm x)^{\pm \frac{m}{n}}$ should be shewn not to exceed any finite ratio; but the investigation of this property, he found, led so directly to the law of the coefficients in each case, that there was no advantage in deducing it separately, in order, afterwards, to form one general demonstration. He could not, however, proceed satisfactorily, in his inquiry into the nature of the series obtained by algebraic evolution, without the help of the Polynomial Theorem. This theorem, therefore, he was obliged previously to demonstrate; the demonstration of it followed, with the greatest ease, from the method which he had employed for the simple involution of the binomial; and it gave him, more than he at first expected to follow immediately from it, namely, the *law* of the coefficients of the expansion of $(a \pm x)^{\frac{1}{n}}$. He has since found that he had been anticipated in this step by Dr. HUTTON.

Complete precision has, undoubtedly, been aimed at in the following proof; the reader will judge whether it has been attained. One objection, however, may probably be foreseen, namely, that some of the conclusions here drawn, may seem not to be arrived at without the help of induction. This term properly denotes the inferring some general proposition from observing that it is true in a multitude of separate instances; no necessary connexion being perceived between the instances themselves and the common pro-

perty. It is thus that the laws of motion are collected, and other axioms in Physics. Induction, in this, the proper sense of the word, is wholly inadmissible in abstract mathematical science. But there is nothing of this kind in the inference, drawn from a partial division of unity by $1-x$, that the $m^{\text{th}}$ term of the quotient must be $x^{m-1}$; the connexion between the subject and the prædicate, the form, and the law of continuation, of the series, is intuitively known; and the mind is as fully satisfied of the truth of the assertion, as it can be of that of any other proposition in Euclid's Elements. The same may be said of the equations which determine the $q^{\text{th}}$ terms of the involution of a binomial, or a polynomial, raised to the $m^{\text{th}}$ power. When a series admits of being continued without limit, as the expansion of $(a \pm x)^{-m}$ and $(a \pm x)^{\pm\frac{m}{n}}$, only a finite nnmber of terms can then be exhibited; but if, in all cases, the *law* of the formation of the terms be found for *any* two successive terms whatever, the $p^{\text{th}}$ and the $(p+1)^{\text{th}}$, it may be fairly concluded to obtain in them all; and this law being once determined, the series may be continued to any number of terms whatever, without Induction.

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ART. 1. DEF. The word *Function* in the following articles, is used to designate any algebraic

expression, containing one or more variable quantities, mixt, or not, with constant quantities; such an expression is called the Function of the variable quantity, or quantities, which it contains.

2. DEF. The *Limit* of a Function is a constant quantity, from which the function may be made to differ less than any other given quantity, but to which it can never be equal.

### PROP. I.

3. *Theorem.* If, in a series of quantities, continued indefinitely, each term be the half of that immediately preceding it, any term whatever is greater than the sum of all the terms which follow it.

Let $K$ be any term whatever of such a series; therefore $\frac{K}{2}$, $\frac{K}{4}$, $\frac{K}{8}$, &c. are the succeeding terms; but, by the common rule, investigated in the Elements of Algebra, $2K$ is the limit of the progression $K + \frac{K}{2} + \frac{K}{4} +$ &c. *ad infinitum*; that is, $2K$ is greater than $K + \frac{K}{2} + \frac{K}{4} +$ &c. to whatever number of terms the series be continued; take $K$ from both, and there remains $K$ greater than $\frac{K}{2} + \frac{K}{4} + \frac{K}{8} +$ &c. *ad infinitum*.

4. Cor. If each term of a series be less than the half of that immediately preceding it, any term whatever is greater than the sum of all the terms which follow it.

## Prop. II.

5. *Theorem.* In a series of quantities, either finite or continued indefinitely, of the form $A \pm Bx + Cx^2 \pm Dx^3 +$ &c., in which the value of $x$ is arbitrary, and the coefficients $A$, $B$, $C$, &c. are constant quantities, none of which has to that preceding it a ratio greater than any ratio assignable, $x$ may be taken such that any term shall be indefinitely greater than the sum of all the terms of the series which contain higher powers of $x$.

Let the constant coefficients $A$, $B$, $C$, &c. be supposed to be all positive, and to go on increasing from $A$, so that the second is greater than the first, the third than the second, and so on; and let $Rx^p$, $Sx^{p+1}$ be the two successive terms in which the ratio of the latter coefficient to the former is greatest; then, since this ratio, by the hypothesis, is finite, $\frac{S}{R}$ is a finite quantity, and, therefore, also $\frac{2 \cdot S}{R}$ is a finite quantity; but the value of $x$ is arbitrary; let, then, $x$ become less

than $\frac{R}{2S}$; therefore $S \,.\, x$ is less than $\frac{R}{2}$, and $S \,.\, x^{p+1}$ less than $\frac{R}{2} \,.\, x^p$; that is, the term $Sx^{p+1}$ is less than the half of the preceding term; but, by the supposition, no other coefficient has to that of the term immediately preceding it, so great a ratio as $S$ has to $R$; wherefore, the second term is less than the half of the first, and every term in the series, after the first, is less than the half of the preceding term; therefore, (Art. 4.) any term of the series is greater than the sum of all the remaining terms; and, if this be the case, when the coefficients are supposed to be all positive, and to go on increasing from $A$, it will, *a fortiori*, be the case, when any other supposition is made, relative to the magnitudes, and the signs of the coefficients; wherefore $x$ may always be so taken, as that any term of the series shall be greater than the sum of all the terms that follow it: and, it is manifest, that by continually diminishing the arbitrary value of $x$, any term may be made to become indefinitely greater than the sum of all the following terms*.

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* All the coefficients, $A$, $B$, $C$, &c., of the series being supposed to have finite and constant values, the proposition may be proved, independently of Art. 4. in the following manner.

First, the value of $x$ may be taken such, that the first term $A$ of the series $A \pm Bx + Cx^2 \pm$ &c. shall have to any other term, as $Q\,x^p$, a ratio greater than any assigned ratio, as that of $r$ to unity.

For,

6. Cor. The first term ($A$) of a series, such as that described in Art. 5, is its limit: and as $A$ may be of any finite value whatever, it is plain that the sum of all the terms, but the first, of such a series

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For, let $s$ denote any number greater than $r$.

And, if $x$, the value of which is arbitrary, be taken equal to $\left(\frac{A}{s \,.\, Q}\right)^{\frac{1}{n}}$,

$$A : Qx^p :: \frac{A}{Q} : x^p$$
$$:: \frac{A}{Q} : \frac{A}{s \,.\, Q}$$
$$.: s : 1.$$

That is, the first term $A$ has to the term $Qx^p$ a greater ratio than that of $r$ to unity, which is any given ratio.

Next, let $m$ denote the number of all the terms of the series but the first, and $t$ any number whatever: let the coefficients be supposed to be all positive and to go on increasing from $A$; and let $x$ be taken so that $A : Bx > mt : 1$; the possibility of which has been demonstrated. Wherefore,

$$\frac{A}{m} : Bx > t : 1;$$
$$\therefore \frac{A}{m} > t \,.\, Bx.$$

Now, it is plain from what has been proved above, and from the supposition made relative to the magnitudes of the coefficients, that $x$ must be taken of a still smaller and smaller value, in order that $A$ may have to $Cx^2$, $Dx^3$, &c. a ratio greater than that of $mt$ to unity: when, therefore, $x$ is taken so that $A$ may have to the last of the terms a ratio greater than that of $mt$ to unity. $\frac{A}{m}$ will still be greater than $t \,.\, Bx$, $t \,.\, Cx^2$, $t \,.\, Dx^3$, and each of the following terms;

$$\therefore \frac{A}{m} + \frac{A}{m} + \text{\&c. } (m \text{ terms}) > (t \,.\, Bx + t \,.\, Cx^2 + t \,.\, Dx^3 + \text{\&c.})$$

i. e.

may be made to become less than any given finite quantity: Further, $A$ is also the limit of a series of the form $A \pm Bx + Cx^2 \pm$ &c. $\pm Rz$, $R$ being a function, either of the arbitrary quantity $z$, or of $x$ and $z$, containing only positive powers, and not containing any coefficient which has, to that of the term immediately preceding it, a ratio greater than any assignable.

For, if $z$, the value of which is arbitrary, become equal to $x$, this series may be made to agree with the series of $A \pm Bx \pm Cx^2 +$ &c., from which it differs only in form; there being no real difference between two perfectly arbitrary quantities, however they may have originated, or by whatever notation they may have been expressed.

## Prop. III.

7. *Theorem.* If two series, of the form described in the last proposition, be equal to each other, the first term of the one is equal to the first

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i. e. $A > t\,(Bx + Cx^2 + Dx^3 + \&c.)$

Wherefore, $Bx + Cx^2 + Dx^3 +$ &c. may be made less than $\frac{A}{t}$, however great $t$ is. Whence it is evident, that the sum of all the terms but the first may be made indefinitely less than any assignable quantity. In the same manner it may be shewn, that $x$ may be so taken, as that *any* term of the series, shall be indefinitely greater than the sum of all the other terms, which contain higher powers of $x$. And if this be true when the coefficients are all positive, and go on increasing from $A$, it will, *a fortiori*, be true in all other cases.

term of the other, and the coefficients of the same powers of the arbitrary quantities, in each, are severally equal.

First, let

$$a \pm bx + cx^2 \pm \&c. \pm qx^p = A \pm Bx + Cx^2 \pm \&c. \pm Qx^p,$$

$$\text{let } \pm bx + cx^2 \pm \&c. \pm qx^p = \pm \pi,$$

$$\text{and } \pm Bx + Cx^2 \pm \&c. \pm Qx^p = \pm \Pi;$$

$$\text{therefore } a \pm \pi = A \pm \Pi.$$

If it be possible, let $a$ and $A$ be unequal, and let their difference be $\delta$, which is, therefore, a finite and constant quantity.

Then $a \sim A = \delta = \Pi \sim \pi$, whatever be the value of $x$; but (Art. 6.) $x$ may be taken such that $\pi$ and $\Pi$ shall each of them be less than any finite quantity; in which case their difference $\Pi \sim \pi$ will manifestly be less than any finite quantity; suppose, therefore, $x$ to be so taken, as that $\Pi$ and $\pi$ shall each of them be less than $\delta$; wherefore $\Pi \sim \pi$ is less than $\delta$; but $\Pi \sim \pi$ is equal to $\delta$; which is absurd; therefore $a = A$; and $\pm bx + cx^2 \pm \&c. \pm qx^p = \pm Bx + Cx^2 \pm \&c. \pm Qx^p$; divide, now, this equation by $x$, and $\pm b + cx \pm \&c. \pm qx^{p-1} = \pm B + Cx \pm \&c. \pm Qx^{p-1}$; $b$ may, therefore, be shewn to be equal to $B$, in the same manner as $a$ was proved to be equal to $A$; and the same may be shewn, in the same manner, of the rest of the coefficients of the corresponding terms of the two series.

8. Cor. 1. If

$$1 \pm bx + cx^2 \pm \text{\&c.} \pm \frac{q\,x^p}{1 + \beta x + \gamma x^2 + \text{\&c.} + \nu x^m} =$$

$$1 \pm Bx + Cx^2 \pm \text{\&c.} \pm \frac{N x^m}{1 + \beta x + \text{\&c.} + \nu x^m};$$

then $1 = 1$, $B = b$, and the rest of the coefficients of the same powers of $x$, in the two series, are equal to each other; the same limitation being laid down with respect to the ratio of any two successive coefficients in the three series, as in Prop. II.

For, let the given equation be multiplied by $1 + \beta x + \text{\&c.} + \nu x^m$.

Then,

$$\left\{ 1 \left.\begin{matrix} \pm b \\ + \beta \end{matrix}\right\} x \left.\begin{matrix} + c \\ \pm \beta b \\ + \gamma \end{matrix}\right\} x^2 \left.\begin{matrix} \pm d \\ + \beta c \\ \pm \gamma b \\ + \delta \end{matrix}\right\} x^3 + \text{\&c.} \right\} =$$

$$= \left\{ 1 \left.\begin{matrix} \pm B \\ + \beta \end{matrix}\right\} x \left.\begin{matrix} + C \\ \pm \beta B \\ + \gamma \end{matrix}\right\} x^2 \left.\begin{matrix} \pm D \\ + \beta C \\ \pm \gamma B \\ + \delta, \end{matrix}\right\} x^3 + \text{\&c.} \right\}$$

$\therefore$ (Art. 7.) $\pm b + \beta = \pm B + \beta$, and $b = B$;

$$c \pm \beta b + \gamma = C \pm \beta B + \gamma;$$

but $B$ has been proved to be equal to $b$;

$\therefore \pm \beta b + \gamma = \pm \beta B + \gamma$, and $c = C$.

In the same manner, $d$ may be shewn to be equal to $D$, and the rest of the coefficients, of the one

series, to the rest of the coefficients, of the other, each to each.

For, if $p$ and $P$ be any two corresponding coefficients, in the two originally given series, the coefficients of the same power of $x$, in the two series resulting from the multiplication, will manifestly be

$$\pm p + \beta . o \pm \gamma . n + \&c. + \pi,$$

$$\text{and} \ \pm P + \beta . O \pm \gamma . N + \&c. + \Pi ;$$

but $\beta . o, \gamma . n$, &c. will have been proved to be equal to $\beta . O, \gamma . N$, &c. respectively; wherefore (Art. 7.) $p = P$.

9. Cor. 2. The same limitation being made, with respect to the coefficients as before, if

$$a \pm bx + cx^2 \pm \&c. \pm lx^k \pm vz =$$
$$A \pm Bx + Cx^2 \pm \&c. \pm Lx^k \pm Vz ;$$

if, also, the values of $x$ and $z$ be both of them arbitrary, and independent of each other; and if $v$ and $V$ be functions of $z$, or of $x$ and $z$, containing only positive powers, then shall

$$a = A, \ b = B, \&c. = \&c. \ l = L.$$

For, first, the series

$$a \pm bx + cx^2 \pm \&c. \pm lx^k,$$

is equal to

$$A \pm Bx + Cx^2 \pm \&c. \pm Lx^k,$$

whatever be the value of $x$: If not, there is some certain value which put for $x$ in the two series will

render them unequal. Let therefore, that value of $x$ be substituted in the given equation, and let the difference between the two series, which involve only powers of $x$ and constant quantities, be denoted by $\delta$:

Therefore, $\delta = (V \sim v)\ z = 0$:

Now this equality is to obtain whatever be the values of $x$ and $z$. But, if the value of $x$ continue to be as it was last assumed, the value of $z$ (Art. 6.) may be so taken as that $(V \sim v)\ z$ shall be less than the constant quantity $\delta$; wherefore, the equality does not obtain, in the equation

$$\delta = (V \sim v)\ z = 0,$$

whatever be the value of $z$; which is contrary to the original supposition. The two series

$$a \pm bx + cx^2 \pm \&c. \pm lx^k,$$
$$A \pm Bx + Cx^2 \pm \&c. \pm Lx^k,$$

cannot, therefore, have any finite difference: that is; they are equal to one another: whence (Art. 7.)

$$a = A,\ b = B,\ c = C,\ \&c. = \&c.\ l = L.$$

## SCHOLIUM.

The value of $x$ in Art. 7. 8. 9. is supposed to be absolutely arbitrary, so that it may be made either greater or less than any finite quantity; but the proposition is true of two such series, which are equal to each other, when $x$ has any particular value, and are also equal, when $x$ has any less value

whatever; even although they would not be equal if $x$ had a value greater than that particular value: the demonstration, indeed, proceeds only upon this latter supposition.

It is, however, implied, in all the cases, that whatever value $x$ has on the one side of the equation

$$a \pm bx + cx^2 + \&c. = A \pm Bx + Cx^2 + \&c.$$

it shall have the same value on the other. In order, therefore, to render the demonstration of Art. 8. unexceptionable, it may be necessary to shew that there is some one value of $x$, which being substituted for it on both sides of the equation, both $\pi$ and $\Pi$ become less than any proposed finite quantity. Now, it is evident from Art. 5. that there is some value of $x$, which renders $\pi$ less than the finite quantity $\delta$: let this value be denoted by $l$. It is evident, likewise, that there is some value of $x$ which renders $\Pi$ less than $\delta$: let that value be denoted by $L$. If, then, either of these two values, as $L$, be greater than the other, it is manifest, that if $l$ be substituted, instead of $L$, for $x$, $\Pi$ will still, *a fortiori*, be less than $\delta$: so that the *same* value of $x$ renders both $\Pi$ and $\pi$ less than the given finite quantity $\delta$.

10. *Definitions.* I. The expansion of $(a \pm x)^m$ is the series arising from the multiplication of $(a \pm x) . (a \pm x) \ldots\ldots$ to $m$ factors.

It is manifest, from the actual process of this

multiplication, that the resulting series is of the form

$$a^m \pm b\,x + c\,x^2 \pm \&c. \pm x^m;$$

and that the coefficients $b$, $c$, &c. are independent of the value of $x$. It is further evident, from the first Principles of Universal Arithmetic, that if two numbers be divided, each into any parts whatever, the product of the one, multiplied by the other, shall be equal to the sum of the products arising from multiplying the parts of the one by the parts of the other; and hence it follows, that, if specific numbers be substituted for $a$ and $x$ in $(a+x)^m$, and in its expansion, already obtained according to any given value of $m$, the number, which is the sum of $a$ and $x$, shall, when raised to the $m^{\text{th}}$ power, be equal to the sum of the terms of the expansion. The sign of *equality* is, therefore, rightly placed between $(a \pm x)^m$ and its expansion, when the index $(m)$ is positive and integral.

The same is true of the series

$$A + Bx + Cx^2 + Dx^3 + \&c. + L\,x^{mk},$$

which arises from the actual multiplication of $(a+bx+cx^2+\&c.+lx^k).(a+bx+cx^2+\&c.+lx^k)$ to $m$ factors, and which is called the expansion of

$$(a + b\,x + c\,x^2 + \&c. + l\,x^k)^m.$$

II. By the expansion of $(a \pm x)^{\frac{1}{n}}$ is meant a series, of the form

$$\alpha \pm \beta x \pm \gamma\,x^2 \pm \delta\,x^3 \pm \&c.$$

such, that if any number $(p + 1)$ of its terms be raised to the $n^{th}$ power, that product is of the form

$$a \pm x + * + * + * \pm Rx^{p+1} \mp \&c. \pm \rho^n x^{np},$$

whatever be the value of $x$. And the expansion of $(a \pm x)^{\frac{m}{n}}$ is the series arising from raising the expansion of $(a + x)^{\frac{1}{n}}$ to the $m^{th}$ power; it is, therefore, of the form

$$A \pm Bx \pm Cx^2 \pm Dx^3 + \&c.$$

in which expression, the coefficients $A, B, C, D,$ &c. are independent of the value of $x$.

III. The expansion of $(a \pm x)^{-m}$ is the series arising from the actual division of unity by the expansion of $(a \pm x)^m$; which series is of the form

$$a' \mp b'x + c'x^2 \mp d'x^3 + \&c.$$

involving only positive powers of $x$, and having its coefficients $a'$, $b'$, $c'$, &c. independent of the value of $x$.

There is, also, an arithmetical equality between

$$\frac{1}{(a \pm x)^m},$$

and $a' \mp b'x + c'x^2 \mp d'x^3 + \&c. \mp \dfrac{Q \cdot x^p}{(a \pm x)^m}$.

This is manifest, from actually dividing 1 by

$$a \mp bx + cx^2 \mp \&c. \mp x^m,$$

which is the expansion of $(a \pm x)^m$, and is also equal to $(a \pm x)^m$.

IV. The expansion of $(a \pm x)^{-\frac{m}{n}}$ is the series arising from the division of unity by the expansion of $(a \pm x)^{\frac{m}{n}}$; and it is, therefore, of the form

$$A' \pm B'x \pm C'x^2 \pm D'x^3 \pm \&c.;$$

and the coefficients $A', B', C', D'$, &c. are also independent of the value of $x$.

## Prop. IV.

11. *Theorem.* The two first terms of the expansion of $(1+x)^m$ are $1 + mx$.

For, Art. 10,

$$(1+x)^m = 1 + bx + cx^2 + dx^3 + \&c. + x^m;$$

$$\therefore (1+x).(1+x)^m = (1+x).(1+bx+cx^2+\&c.+x^m);$$

$$\text{i. e. } (1+x)^{m+1} = 1 + \left.\begin{matrix} b \\ +1 \end{matrix}\right\} x + \left.\begin{matrix} c \\ +b \end{matrix}\right\} x^2 + \&c. + x^{m+1}$$

when, therefore, the index of the power is increased by unity, the coefficient of the second term of the expansion is increased by unity; if, then, the coefficient of the second term of the expansion of $(1+x)^2$ be 2, that of the second term of the expansion of $(1+x)^m$ is $m$; but the coefficient of the second term of the expansion of $(1+x)^2$ is known, from actual multiplication, to be 2; wherefore the coefficient of the second term of the expansion of $(1+x)^m$ is $m$.

12. Cor. 1. The expansion of $(1-x)^m$ is of the form

$$1 - mx + c' x^2 - d' x^3 + \&c. \quad \text{(Art. 10.)}$$

13. Cor. 2. The expansion of $(a+x)^m$ is of the form

$$a^m + m a^{m-1} x + \&c.$$

For, $\left(1 + \frac{x}{a}\right)^m = 1 + m\frac{x}{a} + c\frac{x^2}{a^2} + \&c. + \frac{x^m}{a^m}$.

But $\left(1 + \frac{x}{a}\right)^m = \left(\frac{a+x}{a}\right)^m = \frac{(a+x)^m}{a^m}$.

Therefore,

$$\frac{(a+x)^m}{a^m} = 1 + m\frac{x}{a} + \frac{cx^2}{a^2} + \&c. + \frac{x^m}{a^m};$$

multiply both sides of the equation by $a^m$; then

$$(a+x)^m = a^m + m a^{m-1} x + \&c.$$

In the same manner, it may be shewn that the expansion of $(a+x)^{-m}$ is of the form

$$a^{-m} - m a^{-m-1} x + \&c.;$$

or, that it is the series arising from dividing the expansion of $\left(1 + \frac{x}{a}\right)^m$ by $a^m$.

## Prop. V.

14. *Problem.* To find the expansion of $(a+x)^m$, the index $(m)$ being positive, and a whole number.

It follows, from Art. 10. that,

(A) .... $(1+x)^m = 1 + bx + cx^2 + dx^3 + \&c. + x^m$

and (B) .... $(1+y)^m = 1 + by + cy^2 + dy^3 + \&c. + y^m$.

Let $y$, the value of which is arbitrary, be taken equal to $x+z$, the quantity $z$ having also an arbitrary value; and let $x+z$ be substituted for $y$ in the equation marked (B).

Then $(1+x+z)^m = 1 + b.(x+z) + c.(x+x)^2 + d.(x+z)^3 + \&c. + (x+z)^m =$

$$= \left\{ \begin{array}{l} 1 + bx + cx^2 \quad + dx^3 \quad + \&c. \\ \quad + bz + 2cxz + 3dx^2z \\ \qquad\quad + cz^2 \\ \qquad\qquad\qquad + dz^3 \end{array} \right\} \text{Art. 10. and 11.}$$

But $(1+x+z)^m = \big((1+x)+z\big)^m =$ $(1+x)^m + m.(1+x)^{m-1}z + c.(1+x)^{m-2}z^2 + \&c. + z^m$ (Art. 11.) $\therefore$

(C) .. $(1+x)^m + m(1+x)^{m-1}z + c.(1+x)^{m-2}z^2 + \&c. + z^m =$

$$= \left\{ \begin{array}{l} 1 + bx + cx^2 + dx^3 + \&c. \\ \quad + bz + 2cxz + 3dx^2z + \&c. \\ \qquad\quad + cz^2 \\ \qquad\qquad\qquad + dz^3. \end{array} \right\}$$

Let now the equation (A) be taken from the equation (C), and

$m.(1+x)^{m-1}z + c.(1+x)^{m-2}z^2 + \&c. + z^m =$

$$= \left\{ \begin{array}{l} bz + 2cxz + 3dx^2z + \&c. \\ \qquad + cz^2 \\ \qquad\qquad\qquad + dz^3 \end{array} \right\}$$

Every term of the last equation is divisible by $z$; let it, therefore, be divided by $z$; and

$$m.(1+x)^{m-1} + c(1+x)^{m-2}z + \&c. + z^{m-1} =$$

$$= \left\{ \begin{array}{l} b + 2cx + 3dx^2 + \&c. \\ \quad + cz + \&c. \\ \qquad + dz^2; \end{array} \right\}$$

therefore, (Art. 9.)

$$m.(1+x)^{m-1} = b + 2cx + 3dx^2 + 4ex^3 + \&c.$$

$$\therefore \quad (1+x).m.(1+x)^{m-1} =$$

$$= (1+x).(b + 2cx + 3dx^2 + 4ex^3 + \&c.) \text{ i. e.}$$

$$m.(1+x)^m, \text{ or } m + mbx + mcx^2 + mdx^3 + \&c. + mx^m =$$

$$= \left\{ \begin{array}{l} b + 2cx + 3dx^2 + 4ex^3 + \&c. \\ \quad + bx + 2cx^2 + 3dx^3 + \&c. \end{array} \right\}$$

And, since this equation is true, whatever be the value of $x$, therefore, (Art. 7.)

$m = b$, $2c + b = m.b$, $3d + 2c = m.c$,
$4e + 3d = m.d$, &c. = &c.
$rq + (r-1)p = mp$; therefore

$$c = b.\frac{m-1}{2} = m.\frac{m-1}{2}.$$

$$d = c.\frac{m-2}{3} = m.\frac{m-1}{2}.\frac{m-2}{3}.$$

$$e = d.\frac{m-3}{4} = m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}. \&c. = \&c.$$

$$q = p.\frac{m-(r-1)}{r} = m.\frac{m-1}{2}\ldots\frac{m-(r-1)}{r}.$$

Therefore the expansion of

$$(1+x)^m \text{ is } 1+mx+m.\frac{m-1}{2}x^2+\&c.+x^m,$$

and (Art. 13.), that of

$$(a+x)^m \text{ is } a^m+ma^{m-1}x+m.\frac{m-1}{2}a^{m-2}x^2+\&c.+x^m.$$

15. Cor. 1. The expansion of

$$(a-x)^m \text{ is } a^m-ma^{m-1}x+m.\frac{m-1}{2}a^{m-2}x^2-\&c.\pm x^m.$$

For, it is evident, from the actual involution of $(a-x)^m$, that the coefficients will be the same as those of the expansion of $(a+x)^m$, and that the signs of its terms will be alternately positive and negative.

16. Cor. 2. Since, (Art. 14.) if $b$, $c$, $d$, &c. be the coefficients of the first, second, third, &c. powers of $x$, in the expansion of $(1+x)^m$,

$$(1+x)^m = 1 + bx + \frac{m-1}{2} bx^2 + \frac{m-2}{3} . cx^3 + \frac{m-3}{4} dx^4 + \&c.,$$

and
$$(1+x)^n = 1 + b'x + \frac{n-1}{2} b'x^2 + \frac{n-2}{3} c'x^3 + \frac{n-3}{4} d'x^4 + \&c.;$$

therefore, actually multiplying together the two series,

$$\begin{aligned}(1+x)^{m+n} = 1 + bx &+ \frac{m-1}{2} bx^2 + \frac{m-2}{3} cx^3 + \frac{m-3}{4} dx^4 + \&c.\\
+ b'x &+ b'bx^2 + \frac{m-1}{2} . b'bx^3 + \frac{m-2}{3} b'cx^4 + \&c.\\
&+ \frac{n-1}{2} b'x^2 + \frac{n-1}{2} . b'bx^3 + \frac{n-1}{2} . \frac{m-1}{2} b'bx^4 + \&c.\\
&\qquad + \frac{n-2}{3} c'x^3 + \frac{n-2}{3} . c'bx^4 + \&c.\\
&\qquad\qquad + \frac{n-3}{4} . d'x^4 + \&c.\end{aligned}$$

But also, Art. 14, $(1 + x)^{m+n} =$

$$= 1 + (m + n)x + (m + n) \cdot \frac{(m + n - 1)}{2} \cdot x^2$$

$$+ (m+n) \cdot \frac{(m + n - 1)}{2} \cdot \frac{(m + n - 2)}{3} x^3 + \&c.;$$

therefore, (Art. 7.) $b + b' = m + n$,

$$\frac{m-1}{2} \cdot b + b'b + \frac{n-1}{2} b' = (m+n) \cdot \frac{m+n-1}{2},$$

$$\frac{m-2}{2} \cdot c + \frac{m-1}{2} \cdot b'b + \frac{n-1}{2} \cdot b'b + \frac{n-2}{3} \cdot c' =$$

$$= (m + n) \cdot \frac{m+n-1}{2} \cdot \frac{m+n-2}{3};$$

and the rest of the coefficients of the different powers of $x$ in the product of the two series multiplied together, are severally equal to the coefficients of the same powers of $x$ in the expansion of $(1 + x)^{m+n}$; therefore the former set of coefficients are reducible to the form of the latter: and that they admit of being reduced to this *form* does not at all depend upon $m$ and $n$ being integers. If $\frac{p}{q}$ were put for $m$, and $\frac{r}{s}$ for $n$, in the two series multiplied together, the resulting coefficients would be of the same *form* as before, and might, therefore, be resolved in the same manner; so that if

$$1+\frac{p}{q}x+\frac{p}{q}\left\{\frac{\frac{p}{q}-1}{2}\right\}x^2+\frac{p}{q}\left\{\frac{\frac{p}{q}-1}{2}\right\}.\left\{\frac{\frac{p}{q}-2}{3}\right\}x^3+\&c.$$

be multiplied by

$$1+\frac{r}{s}x+\frac{r}{s}.\left\{\frac{\frac{r}{s}-1}{2}\right\}x^2+\frac{r}{s}\left\{\frac{\frac{r}{s}-1}{2}\right\}.\left\{\frac{\frac{r}{s}-2}{3}\right\}x^3+\&c.$$

the several coefficients of the different powers of $x$ in the product, are reducible to the form

$$\frac{p}{q}+\frac{r}{s},\ \left(\frac{p}{q}+\frac{r}{s}\right).\left\{\frac{\frac{p}{q}+\frac{r}{s}-1}{2}\right\},\ \&c.$$

And (Art. 13.) if

$$a^{\frac{p}{q}}+\frac{p}{q}a^{\frac{p}{q}-1}x+\frac{p}{q}.\frac{\frac{p}{q}-1}{2}a^{\frac{p}{q}-2}x^2+\&c.$$

be multiplied by

$$a^{\frac{r}{s}}+\frac{r}{s}.a^{\frac{r}{s}-1}x+\frac{r}{s}.\left\{\frac{\frac{r}{s}-1}{2}\right\}a^{\frac{r}{s}-2}+\&c.$$

the resulting series will be of the form

$$a^{\frac{p}{q}+\frac{r}{s}}+\left(\frac{p}{q}+\frac{r}{s}\right)a^{\frac{p}{q}+\frac{r}{s}-1}x+\&c.$$

## SCHOLIUM.

If the problem solved in Art. 14. had been the limit of our enquiries, in this part of the subject, a plainer course would have been chosen, than that which has there been pursued. The mode of demonstration actually employed in that article, has, indeed, been adopted, for the sake of giving uniformity to the whole investigation of the form of the expansion of $(a+x)^{\pm\frac{m}{n}}$, in all its several cases.

The following proof of the Binomial Theorem, in the case of a positive and integral index, is perhaps less circuitously derived from first principles, than any other. It is founded principally on the Doctrine of Permutations; than which no part of Algebra is more free from difficulty and obscurity.

The problem is, to determine the form of the product of $(a+x) \cdot (a+x) \cdot (a+x)$, &c. when there are $m$ factors.

Let the multiplication be supposed to proceed, without any algebraic addition of the several partial products thence successively arising; putting, always, the letter which is the multiplier before that which is the multiplicand. Thus,

$$\begin{aligned}(a+x)^1 &= a+x \\ (a+x)^2 &= (a+x) \cdot (a+x) \\ &= aa + ax + xa + xx.\end{aligned}$$

$$\begin{aligned}(a+x)^3 &= (a+x)^2 . (a+x) \\ &= (aa+ax+xa+xx) . (a+x) \\ &= aaa+aax+axa+axx+xaa+ \\ &+ xax+xxa+xxx, \\ \&c. &= \&c.\end{aligned}$$

Thus, the process of multiplication consists, entirely, in placing, at each step, first $a$ and then $x$ before each of the permutations, which, together, form the next preceding product; and thus, it is very evident, that the square consists of all the different permutations that can be made out of $aaxx$, taken two by two; the cube, of all that can be made out of $aaaxxx$, taken three by three; and so on; the $m^{\text{th}}$ power, of all that can be made out of $aaa \ldots (m)$ $bbb \ldots (m)$ taken $m$ together. It is, also, equally manifest, that the partial products, or several collections, each containing $m$ of those $2m$ quantities will be of the forms,

$$a^m,\ a^{m-1}x,\ x^{m-2}x^2,\ \&c.\ \&c.\ a^2x^{m-2},\ ax^{m-1},\ x^m,$$

and that there will be as many products, of any one of those forms, as there are permutations of the letters which compose it. The form $a^{m-3}x^3$, for example, will occur as often as there are different permutations of the letters $aaa \ldots (m-3)\ xxx$, taken $m$ together. Hence, from the well-known theory of permutations,

$$(a+x)^m = a^m + m a^{m-1}x + m . \frac{m-1}{2} a^{m-2}x^2 + \&c.\ \&c. +$$

$$m . \frac{m-1}{2} a^2x^{m-2} + m . ax^{m-1} + x^m.$$

COR. The *number* of all the terms, in the product of

$$(a + x) \,.\, (a + x) \,.\, (a + x) \ldots\ldots (m),$$

when no addition has been made of the several partial products successively arising, is equal to $2^m$: for it is 4, or $2^2$, in the product of $(a + x) \,.\, (a + x)$, and it is doubled at every succeeding step. It is evident, therefore, that the sum of the coefficients, in this case, is equal to $2^m$; for each coefficient is unity; so that their sum is equal to the number of terms, that is, to $2^m$. And, as the sum of the coefficients, in this dilated form of the product, is necessarily equal to the sum of the coefficients in that more compact form of it, which is expressed by the Binomial Theorem, therefore,

$$2^m = m + m \,.\, \frac{m-1}{2} + m \,.\, \frac{m-1}{2} \,.\, \frac{m-2}{3} + \&c.$$

In the same manner may be investigated the expansion of the polynomial

$$(a + b + c + d + \&c.)^m,$$

when the index $m$ is integral and positive.

Thus the demonstration of the Binomial and Polynomial Theorems, in the simplest case of them, is made to depend on the solution of this question: "If there be $p$ quantities all of one kind, and $q$ quantities all of another kind, and if $p + q = m$, what is the number of permutations of which these quantities admit, taken $m$ together?" To this question the following theorem, deducible from

the common rules relating to combinations and permutations, affords a ready answer.

If $M$ be put for the number of permutations of which $m$ quantities admit, taken all together, $P$ for the number of permutations of which $p$ quantities admit, and $Q$ for the number of which $q$ quantities admit; if $m = p + q$; and if $C$ be put for the number of combinations of which $m$ quantities admit, when $p$ or $q$ of them are taken together, then,

$$M = P \times Q \times C.$$

For, (*Wood's* Algebra, Art. 229, 230.)

$$M = m \,.\, (m-1) \,.\, (m-2) \ldots\ldots 1.$$

$$P = p \,.\, (p-1) \,.\, (p-2) \ldots\ldots 1.$$

$$Q = q \,.\, (q-1) \,.\, (q-2) \ldots\ldots 1.$$

$$C = \frac{m \,.\, (m-1) \ldots\ldots [m-(q-1)]}{1 \,.\, 2 \ldots\ldots\ldots\ldots q};$$

$$\therefore M = P \times Q \times C.$$

Hence, if all the $p$ quantities be of one kind, and the $q$ quantities be all of another kind,

$$P = 1; \quad Q = 1;$$

$$\text{and } M = P \times Q \times C =$$

$$\frac{m \,.\, (m-1) \,.\, (m-2) \ldots [m-(q-1)]}{1 \,.\, 2 \,.\, 3 \qquad\qquad q};$$

which, therefore, is the general form of the coefficient of the $q^{th}$ term of the expansion of $(a+x)^m$.

In the same manner it may be shewn, in general, that if

$$p + q + r + \&c. = m,$$

and if there be $p$ quantities of one kind, $q$ of another, $r$ of a third kind, and so on, the number of permutations of these quantities, taken $m$ together, is equal to

$$\frac{m\,.\,(m-1)\,(m-2)\ldots\ldots\ldots 1}{(1\,.\,2\ldots .p)\,.\,(1\,.\,2\ldots .q)\,.\,(1\,.\,2\ldots .r)\ \&c.}.$$

## Prop. VI.

17. *Theorem.* If $A, B, C, D$, &c. be a series of quantities, such that

$$B = mA,\ C = \frac{m \pm 1}{2}\,.\,B,$$

$$D = \frac{m \pm 2}{3}\,.\,C,\ \&c. = \&c.\ Q = \frac{m \pm (p-1)}{p}\,.\,P;$$

then $A \pm B + C \pm \&c. \pm P = \pm \dfrac{p\,.\,Q}{m}$.

For $A = \dfrac{B}{m}$, and, therefore,

$$A \pm B = \frac{B}{m} \pm B = \frac{1 \pm m}{m}\,.\,B = \pm \frac{2\,C}{m}.$$

Hence,

$$A \pm B + C = C \pm \frac{2\,C}{m} = \frac{m \pm 2}{m}\,C = \frac{3\,D}{m};$$

and thus it is evident, that

$$A \pm B + C \pm \&c. \pm P = \pm \frac{p\,.\,Q}{m},$$

if $p$ terms be added together.

18. Cor. Hence,

$2C + m(B + A) = 4C$; $3D + m(C + B + A) = 6D$; &c. = &c.; $rS + m(R + Q + P + \text{&c.} + A) = 2rS$.

## Prop. VII.

19. *Problem.* To find the expansion of

$$(\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{&c.} + \sigma x^r)^m,$$

the index $m$ being positive, and a whole number.

Let, Art. 10.,

$$(\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{&c.} + \sigma x^r)^m =$$
$$A + Bx + Cx^2 + Dx^3 + \text{&c.} + \sigma^m x^{mr},$$

and $(\alpha + \beta y + \gamma y^2 + \delta y^3 + \text{&c.} + \sigma y^r)^m =$
$$A + By + Cy^2 + Dy^3, \text{&c.} + \sigma^m y^{mr};$$

and, the value of $y$ being arbitrary, let $y = x + z$; then,

$$\left\{\begin{array}{llll} \alpha + \beta x + & \gamma x^2 + \delta x^3 + \text{&c.} & + \sigma x^r \\ \quad + \beta z + & 2\gamma x z + 3\delta x^2 z & + r\sigma x^{r-1} z \\ & \quad + \text{&c.} + \text{&c.} & + \text{&c.} \\ & & + z^r. \end{array}\right\}^m =$$

$$= \left\{\begin{array}{llll} A + Bx + Cx^2 + Dx^3 + \text{&c.} + \sigma^m x^{mr} \\ \quad + Bz + 2Cxz + 3Dx^2 z + mr.\sigma^m x^{mr-1} z \\ \qquad\qquad + \text{&c.} \qquad + \text{&c.} \qquad + \text{&c.} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sigma^m z^{mr}; \end{array}\right\}$$

that is, if $S$ be put for

$$\alpha + \beta x + \gamma x^2 + \text{&c.} + \sigma x^r,$$

and $Vz$ for the rest of the terms of the first

member of the equation, all of which contain $z$ as a factor,

$$(S+Vz)^m = \left.\begin{array}{l} A + Bx + Cx^2 + Dx^3 + \&c. \\ \qquad + Bz + 2Cxz + 3Dx^2z \\ \qquad\qquad + \&c. \quad + \quad \&c. \end{array}\right\}$$

therefore, (Art. 13.)

$$S^m + mS^{m-1}Vz + cS^{m-2}V^2z^2 + \&c. =$$

$$= \left\{\begin{array}{l} A + Bx + Cx^2 + Dx^3 + \&c. \\ \quad + Bz + 2Cxz + 3Dx^2z \\ \qquad\quad + Cz^2 + \&c. \\ \qquad\qquad\quad + Dz^3 \end{array}\right\}$$

But $S^m = A + Bx + Cx^2 + Dx^3 + \&c.$

therefore,

$$mS^{m-1}Vz + cS^{m-2}V^2z^2 + \&c. =$$

$$= \left\{\begin{array}{l} Bz + 2Cxz + 3Dx^2z + 4Ex^3z + \&c. \\ \qquad + Cz^2 + \&c. + \&c. \\ \qquad\qquad + Dz^3 + Ez^4. \end{array}\right\}$$

Every term of this last equation is divisible by $z$; let it, therefore, be divided by $z$, and,

$$mS^{m-1}V + cS^{m-2}V^2z + \&c. =$$

$$= \left\{\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + \&c. \\ \qquad + Cz \quad + \&c. \quad + \&c. \\ \quad + Dz^2 \\ \qquad\qquad\qquad\qquad + Ez^3 \end{array}\right\}$$

Now the only part of $V$, which does not contain any power of $z$, is

$$\beta + 2\gamma x + 3\delta x^2 + \&c. + r\sigma x^{r-1};$$

therefore, (Art. 9.)

$$mS^{m-1}\,.\,(\beta + 2\gamma x + 3\delta x^2 + \&c. + r\sigma x^{r-1}) = (B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.)$$

$$\therefore\ mS^m(\beta + 2\gamma x + 3\delta x^2 + \&c. + r\sigma x^{r-1}) = S\,.\,(B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.)$$

i. e. $(mA + mBx + mCx^2 + mDx^3 + \&c.)\,.\,(\beta + 2\gamma x + 3\delta x^2 + \&c.) =$

$= (\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.)\,.\,(B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.)$

Hence, the two series, in each case, being multiplied together,

$$m\beta.A + \left.\begin{matrix} m\beta.B \\ +2m\gamma.A \end{matrix}\right\}.x + \left.\begin{matrix} m\beta.C \\ +2m\gamma.B \\ +3m\delta.A \end{matrix}\right\}.x^2 + \left.\begin{matrix} m\beta.D \\ +2m\gamma.C \\ +3m\delta.B \\ +4m\epsilon.A \end{matrix}\right\}.x^3 + \left.\begin{matrix} m\beta.E \\ +2m\gamma.D \\ +3m\delta.C \\ +4m\epsilon.B \\ +5m\zeta.A \end{matrix}\right\}.x^4 + \&c. =$$

$$= \alpha.B + \left.\begin{matrix} 2\alpha.C \\ +\ \beta.B \end{matrix}\right\}.x + \left.\begin{matrix} 3\alpha.D \\ +2\beta.C \\ +\ \gamma.B \end{matrix}\right\}.x^2 + \left.\begin{matrix} 4\alpha.E \\ +3\beta.D \\ +2\gamma.C \\ +\ \delta.B \end{matrix}\right\}.x^3 + \left.\begin{matrix} 5\alpha.F \\ +4\beta.E \\ +3\gamma.D \\ +2\delta.C \\ +\ \epsilon.B \end{matrix}\right\}.x^4 + \&c.$$

Wherefore, (Art. 7.)

$$\alpha . B = m\beta . A$$

$$2\alpha . C = (m-1)\beta . B + 2m\gamma . A$$

$$3\alpha . D = (m-2)\beta . C + (2m-1) . \gamma . B + 3m\delta A$$

$$4\alpha . E = (m-3)\beta . D + (2m-2) . \gamma . C + \\ + (3m-1)\delta . B + 4m\epsilon . A$$

$$5\alpha . F = (m-4)\beta . E + (2m-3) . \gamma . D + \\ (3m-2)\delta . C + (4m-1) . \epsilon . B + 5m\zeta . A$$

&c. = &c.

$$p\alpha . Q = (m-(p-1))\beta . P + (2m-(p-2))\gamma . O + \&c. \\ + ((p-1)m-1)\pi . B + pm\rho . A,$$

$Q$ being the coefficient of the $(p+1)^{\text{th}}$ term.

Now it is manifest, that $A = a^m$;

$$\therefore B = m . \frac{\beta . a^m}{a} = m a^{m-1}\beta.$$

$$C = m . \frac{m-1}{2} a^{m-2}\beta^2 + m a^{m-1}\gamma.$$

$$D = m . \frac{m-1}{2} . \frac{m-2}{3} a^{m-3}\beta^3 + \\ + m . \frac{m-1}{1} a^{m-2}\beta\gamma + m a^{m-1}\delta,$$

&c. = &c.

and, all the assumed coefficients having been thus investigated, the expansion of the polynomial is found.

20. Cor. 1. If $\beta$, $\delta$, $\zeta$, &c. the coefficients of the alternate terms of

$$a + \beta x + \gamma x^2 + \delta x^3 + \&c.$$

be negative, it is manifest, from the equations last

obtained, that the values of $B$, $D$, $F$, &c., the coefficients of the alternate terms in the expansion, will also be wholly negative.

21. Cor. 2. The first $(p+1)$ terms of the expansions of

$$(\alpha + \beta x + \gamma x^2 + \&c. + \rho x^p)^m,$$

and of

$$(\alpha + \beta x + \gamma x^2 + \&c. + \rho x^p + \sigma x^{p+1} + \&c.)^m$$

will be the same. For the same equations determine the coefficients $Q$, $P$, &c. in both cases.

22. Cor. 3. If $\alpha$, $\beta$, $\gamma$, &c. be each unity, the equations which determine the coefficients of the expansion of

$$(1 + x + x^2 + \&c.)^m$$

become

$$B = m \,.\, A = m$$
$$2C = (m-1) \,.\, B + 2B = (m+1) \,.\, B$$
$$3D = (m-2) \,.\, C + 2C + m \,.\, (B + A) =$$
$$= (m-2) \,.\, C + 4C = (m+2) \,.\, C \text{ (Art. 17.)}$$
$$4E = (m-3) \,.\, D + 3D + m \,.\, (C + B + A) =$$
$$= (m-3) \,.\, D + 6D = (m+3) \,.\, D,$$
$$\&c. = \&c.$$

Whence,

$$(1 + x + x^2 + x^3 + \&c.)^m = 1 + mx +$$
$$+ \frac{m+1}{2} \,.\, Bx^2 + \frac{m+2}{3} \,.\, Cx^3 + \frac{m+3}{4} \,.\, Dx^4 + \&c.;$$

and (Art. 20.)

$$(1-x+x^2-x^3+\&c.)^m = 1 - mx + \frac{m+1}{2}\,.\,Bx^2 -$$

$$\frac{m+2}{3}\,.\,Cx^3 + \frac{m+3}{4}\,.\,Dx^4 - \&c.$$

23. Cor. 4. The expansion of $(a+x)^{\frac{1}{n}}$ is of the form

$$a^{\frac{1}{n}} + \frac{1}{n}a^{\frac{1}{n}-1}x + \frac{1}{n}\,.\left\{\frac{\frac{1}{n}-1}{2}\right\}a^{\frac{1}{n}-2}x^2 + \&c.$$

in which the law of formation of the coefficients is the same as that in the expansion of $(a+x)^n$.

For, let

$$a + \beta x + \gamma x^2 + \&c. + \rho x^p$$

be the $(p+1)$ first terms of the expansion of $(a+x)^{\frac{1}{n}}$; then, (Art. 10. Def. II.),

$(a + \beta x + \gamma x^2 + \&c. + \pi x^{p-1} + \rho x^p)^n =$
$= a + x + * + * + \&c. + R x^{p+1} + \&c. + \rho^n x^{pn}.$

And, comparing this with the equation which determines the coefficients of the $(p+1)^{\text{th}}$ term of the expansion of a polynomial, $A=a$, $B=1$, $C$, $D$, &c. $P$, $Q$, are each equal to nothing; whence
$pa \times 0 = 0 + 0 + \&c. + ((p-1)n-1)\,.\,\pi \times 1 + pn\rho\,.\,a.$

$$\therefore np\rho\,.\,a = (1-(p-1)\,.\,n)\,.\,\pi, \text{ and } \rho = \frac{\frac{1}{n}-(p-1)}{p\,.\,a}\,.\,\pi;$$

but $\rho$ and $\pi$ are any consecutive coefficients of the expansion of $(a+x)^{\frac{1}{n}}$; wherefore the law of the

formation of the coefficients is the same as that of the coefficients of $(a+x)^m$, substituting $\frac{1}{n}$ for $m$ (Art. 14.); and the expansion of $(a+x)^{\frac{1}{n}}$ is $a^{\frac{1}{n}} +$

$$\frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{ \frac{\frac{1}{n} - 1}{2} \right\} a^{\frac{1}{n}-2} x^2 + \&c.$$

24. Cor. 5. The expansion of $(a+x)^{\frac{1}{n}}$ is the same as the series obtained by treating $a + x$ according to the rule for algebraically extracting the $n^{th}$ root of a compound quantity; that is, by taking $a^{\frac{1}{n}}$ for the first term of the series, as being the $n^{th}$ root of $a$, subtracting $\left(a^{\frac{1}{n}}\right)^n$, or $a$, from $a + x$; dividing the first term of the remainder by $n a^{\frac{n-1}{n}}$, and making the quotient the second term of the series; which will, therefore, be

$$\frac{1}{n} \cdot \frac{x}{a^{\frac{n-1}{n}}}, \text{ or } \frac{1}{n} a^{\frac{1}{n}-1} x;$$

then subtracting from $a + x$ the $n^{th}$ power of the terms already found, and repeating the same process.

For the three first terms thus obtained being

$$a^{\frac{1}{n}} + \beta x + \gamma x^2,$$

it follows from Article 21. that

$$\left(a^{\frac{1}{n}} + \beta x + \gamma x^2\right)^n = a + x + 0 + D x^3 + \&c.;$$

and (Art. 19.)

$$3\,a^{\frac{1}{n}}\,.\,D = (n-2)\,.\,\beta \times 0 + (2n-1)\,.\,\gamma\,;$$

therefore $D = \dfrac{2n-1}{3} \cdot \dfrac{\gamma}{a^{\frac{1}{n}}}$; and, by the rule, the next term of the series is

$$\frac{-D}{n\,a^{\frac{n-1}{n}}}\,, \text{ or } \left\{\frac{\frac{1}{n}-2}{3}\right\}\cdot\frac{\gamma}{a}$$

as before. In the same manner, the remaining terms of the series obtained by this rule, may be shewn to be identical with those of the expansion of $(a+x)^{\frac{1}{n}}$. And thus, after $p+1$ terms have been found

$$\left(a^{\frac{1}{n}} + \beta x + \gamma x^2 + \&\text{c. } \pi x^{p-1}\right)^n =$$

$$a + x + 0 + 0 + \&\text{c.} + Q x^p + \&\text{c. } \pi^n x^{n(p-1)}:$$

whence,

$$p\,a^{\frac{1}{n}}\,.\,Q = ((p-1)\,.\,n - 1)\,.\,\pi\,;$$

therefore,

$$Q = \frac{(p-1)\,.\,n-1}{p}\cdot\frac{\pi}{a^{\frac{1}{n}}}; \text{ and}$$

$$\rho = \frac{-Q}{n\,a^{\frac{n-1}{n}}} = \frac{\frac{1}{n}-(p-1)}{p}\cdot\frac{\pi}{a}.$$

In the same manner it may be shewn that the expansion of

$$(a-x)^{\frac{1}{n}} \text{ is } a^{\frac{1}{n}} - \frac{1}{n} a^{\frac{1}{n}-n} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n}-1}{2}\right\} a^{\frac{1}{n}-2} x^2 - \&c.$$

25. Cor. 6. The expansion of

$$(a+x)^{\frac{m}{n}} \text{ is } a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m}{n}-1} x + \frac{m}{n} \cdot \left\{\frac{\frac{m}{n}-1}{2}\right\} a^{\frac{m}{n}-2} x^2 + \&c.;$$

in which series the coefficients are formed according to the same law as those of the expansion of $(a+x)^m$.

For the expansion of $(a+x)^{\frac{m}{n}}$ is (Art. 10. and 23.) the product of

$$\left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n}-1}{2}\right\} a^{\frac{1}{n}-2} x^2 + \&c.\right\}$$

$$\times \left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n}-1}{2}\right\} a^{\frac{1}{n}-2} x^2 + \&c.\right\} \times \&c.$$

to $m$ factors: but (Art. 16.) the product of

$$\left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n}-1}{2}\right\} a^{\frac{1}{n}-2} x^2 + \&c.\right\} \times$$

$$\left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n}-1}{2}\right\} a^{\frac{1}{n}-2} x^2 + \&c.\right\}$$

is $a^{\frac{2}{n}} + \frac{2}{n} . a^{\frac{2}{n}-1} x + \frac{2}{n} . \left\{\frac{\frac{2}{n} - 1}{2}\right\} a^{\frac{2}{n}-2} x^2 + \&c.$;

wherefore, also, by the same Article,

$$\left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \&c.\right\}^3$$

is

$$a^{\frac{3}{n}} + \frac{3}{n} a^{\frac{3}{n}-1} x + \frac{3}{n} . \left\{\frac{\frac{3}{n} - 1}{2}\right\} a^{\frac{3}{n}-2} x^2 + \&c.;$$

and

$$\left\{a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \&c.\right\}^m$$

is

$$a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m}{n}-1} x + \frac{m}{n} . \left\{\frac{\frac{m}{n} - 1}{2}\right\} a^{\frac{m}{n}-2} x^2 + \&c.$$

And it is manifest (from Art. 20. and 23.) that the expansion of $(a - x)^{\frac{m}{n}}$ is the same, with the exception of its even terms having the negative, instead of the positive, sign.

26. Cor. 7. If $m = n$ the expansion of $(a + x)^{\frac{m}{n}}$, becomes $a + x$, or the $m^{\text{th}}$ root of $(a + x)^m$; for every term, after the two first terms of the series, then vanishes, and the two first terms become $a + x$. Also, if $S$ be put for the sum of any

number $(p+1)$ of the first terms of the series,

$$a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{1}{n} \cdot \left\{\frac{\frac{1}{n} - 1}{2}\right\} a^{\frac{1}{n}-2} x^2 + \&c.$$

and $\alpha$, $\beta$, $\gamma$, $\delta$, &c. be put for the first term, and the successive coefficients of the powers of $x$ in the expansion of $(a+x)^{\frac{m}{n}}$, then,

$$S.S.\&c.\ (m) = \alpha + \beta x + \gamma x^2 + \&c. + q^m x^{mp};$$

$\therefore (S.S\ldots(m)).(S.S\ldots(m)).$ &c. to $n$ factors, $=$ $(\alpha+\beta x+\gamma x^2+\&c.)\times(\alpha+\beta x+\gamma x^2+\&c.)\ \&c.\ (n)$ i.e. $(S.S\ldots(n)).(S.S\ldots(n)).$ &c. to $m$ factors $=$ $(\alpha+\beta x+\gamma x^2+\&c.)\times(\alpha+\beta x+\gamma x^2+\&c.)\ \&c.\ (n)$, because (Euclid 16. 7.) $m$ times $n$ is equal to $n$ times $m$. Wherefore (Art. 10. Def. II.)

$$(a + x + * + * + \&c. + Rx^{p+1} + \&c. + \rho^n x^{np})^m =$$
$$= (\alpha + \beta x + \gamma x^2 + \&c. + q^m x^{mp})^n :$$

whence, the expansion of $(a+x)^{\frac{m}{n}}$ is a series, such that if any number $(p+1)$ of its terms be raised to the $n^{\text{th}}$ power, that product is of the form

$$(a + x)^m + 0 + 0 + \&c. + ma^{m-1} Rx^{p+1} + \&c.$$

For, (Art. 21.) the first $p+1$ terms of the expansion of

$$(\alpha + \beta x + \gamma x^2 + \&c.)^n$$

will be the same, whether $p+1$ terms, or more, be taken. The expansion of $(a+x)^{\frac{m}{n}}$ is, therefore, the

same as that obtained by the rule for extracting the $n^{th}$ root of a compound algebraical quantity.

27. Cor. 8. The expansion of $(a+x)^{-m}$ is

$$a^{-m} - m\frac{a'}{a}.x + \frac{m+1}{2}.\frac{b'}{a}x^2 - \frac{m+2}{3}.\frac{c'}{a}x^3 + \&c. ..$$

in which series the coefficients are formed according to the same law as those of the expansion of $(a+x)^m$; $a'$ being the first term, $b'$ the whole coefficient of $x$ in the second, $c'$ that of $x^2$ in the third, &c.

For, (Art. 10. Def. III.)

$$\frac{1}{(1+x)^m} = 1 - b'x + c'x^2 - d'x^3 + \&c. \pm \frac{Q.x^p}{(1+x)^m}\text{*}.$$

Also,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c. \pm \frac{x^k}{1+x}.$$

But

$$\frac{1}{1+x}.\frac{1}{1+x}\ \&c.\ (\text{to } m \text{ factors}) = \frac{1}{(1+x)^m};$$

---

* If $b, c, d, e$, &c. be the coefficients of the powers of $x$, in the expansion of $(1+x)^m$, and unity be actually divided by that expansion, it will be found that

$b' = b$
$c' = b'b - c$
$d' = c'b - b'c + d$
&c. = &c.
$l' = k'b - i'c + h'd -$ &c. $\pm l$.

Which equations contain the law of the formation of the coefficients in the expansion of $(1+x)^{-m}$, although it cannot readily be deduced from them.

$$\therefore \left\{1 - x + x^2 - \&c. \pm \frac{x^k}{1+x}\right\}^m =$$

$$= 1 - b'x + c'x^2 - d'x^3 + \&c. \pm \frac{Q \cdot x^p}{(1+x)^m}.$$

But (Art. 22.)

$$\left\{1 - x + x^2 - \&c. \pm \frac{x^k}{1+x}\right\}^m =$$

$$= 1 - mx + \frac{m+1}{2} \cdot Bx^2 - \frac{m+2}{3} \cdot Cx^3 +$$

$$+ \&c. \pm \frac{x^{mk}}{(1+x)^m}; \ B, C, \&c.$$

being put for the coefficients of $x^2$, $x^3$, &c.;

$\therefore$ (Art. 8.) $b' = B$; $c' = C$; $d' = D$; &c. = &c.

Wherefore, the expansion of

$(1+x)^{-m}$ is $1 - mx + \frac{m+1}{2} \cdot b'x^2 - \frac{m+2}{3} c'x^3 +$ &c.

And, by substituting $\frac{x}{a}$ for $x$ in the equation last investigated, the expansion of $(a+x)^{-m}$ may be shewn to be

$$a^{-m} - \frac{ma'}{a} x + \frac{m+1}{2} \cdot \frac{b'}{a} x^2 - \frac{m+2}{3} \cdot \frac{c'}{a} x^3 + \&c.:$$

Also, since

$$\frac{1}{1-x} = 1 + x + x^2 + \&c. + \frac{x^k}{1+x}$$

the expansion of $(a - x)^m$ may in the same manner be shewn to be

$$a^{-m} + m\frac{a'}{a}x + \frac{m+1}{2}\cdot\frac{b'}{a}x^2 + \frac{m+2}{3}\cdot\frac{c'}{a}x^3 + \&c.$$

28. Cor. 9. The expansion of $(a \pm x)^{\frac{-m}{n}}$ is $a^{\frac{-m}{n}}$

$$\mp \frac{m}{n}A'x + \frac{\frac{m}{n}+1}{2}\cdot\frac{B'}{a}\cdot x^2 \mp \frac{\frac{m}{n}+2}{3}C'x^3 + \&c.$$

For (Art. 26.) when unity is divided by

$$a^m \pm m a^{m-1}x + m\cdot\frac{m-1}{2}\cdot x^2 \pm \&c.$$

the coefficients of the different powers of $x$ in the quotient can be reduced to the form

$$m,\ m\cdot\frac{m+1}{2},\ m\cdot\frac{m+1}{2}\cdot\frac{m+2}{3},\ \&c.$$

And when unity is divided by

$$a \pm \frac{m}{n}x + \frac{m}{n}\cdot\left\{\frac{\frac{m}{n}-1}{2}\right\}\cdot x^2 \pm \&c.,$$

the coefficients of the several powers of $x$ in the quotient will be the same as before, except that $\frac{m}{n}$ must be written for $m$; wherefore, these coefficients may be reduced to the form

$$\frac{m}{n},\ \frac{m}{n}\cdot\left\{\frac{\frac{m}{n}+1}{2}\right\},\ \&c.$$

29. Cor. 10. Thus, whether the index $m$ of $(a+x)^m$ be positive or negative, integral or fractional, it may be shewn, that the coefficients of the

expansion of $(a+x)^m$ are formed according to the same law; and, that in no case of the expansion of a binomial, is the ratio of the latter of two consecutive coefficients, to the former, greater than any that can be assigned: And, therefore, the value of $x$ may be taken such, that the difference between any one term, and the sum of all that follow it, shall be less than any given finite quantity.

For, let $p$ and $q$ be any two consecutive coefficients; it has been demonstrated, in every case of the expansion of $(a+x)^m$, that

$$\pm\, q = \frac{\pm\, m \mp (r-1)\,.\,p}{r};$$

$$\text{therefore } \frac{q}{p} = \frac{\pm\, m \mp (r-1)}{r};$$

which quantity is always less than $\dfrac{m+(r-1)}{r}$;

and, if $r$ be greater than $m$, it is less than 2. Therefore the ratio of $q$ to $p$ can never be so great as that no ratio can be assigned which is equally great; and, therefore, (Art. 6.) the value of $x$ may be taken such, as that any one term of the expansion shall become the limit of all the terms which follow it.

ON

# MAXIMA AND MINIMA.

## PART II.

### SECTION II.

ON THE EQUATION WHICH SERVES TO DETERMINE THE VALUE OF ANY FUNCTION OF ONE OR MORE VARIABLE QUANTITIES, WHEN IT IS A MAXIMUM OR A MINIMUM.

PREVIOUSLY to the estimation of continued quantity, it is necessary to make some hypothesis respecting the generation of variable magnitudes.

BARROW enumerates eight different modes in which quantity may be supposed to be generated. Its increase and decrease by motion, which is the foundation of the doctrine of Fluxions, is readily conceived in a vague and general manner. But there is no inconsiderable difficulty in deducing,

logically, from that primary notion, the rules of algebraic computation, without which mere theory is of little value. Motion implies velocity; velocity requires the consideration of time; and to any enquiry concerning the nature of time we are not yet enabled to return a much more satisfactory answer than that of AUGUSTIN, so often cited, "*Si nemo quærat, scio; si quis interroget, nescio.*"

All that seems necessary, in the first instance, in the place of the fluxional hypothesis, is to express, in as general a manner as is possible, the condition of a quantity being variable according to a certain law; i. e. of its admitting any variation whatever with respect to magnitude, so as to continue to be the same species of magnitude, as it was before that variation took place. If $z$ denote a line variable at pleasure, and $z'$ any addition or diminution of which it is capable, then $z \pm z'$ will denote the condition of that line: and if $az^m$ denote a surface, or a solid, variable at pleasure, but always retaining its species, expressed in terms of a variable line $z$, $a(z \pm z')^m$ will be a general representative of its value. Of this kind is the notation adopted in the following section.

There is little real difference, in the methods used by the ancient and modern authors, who have treated the investigation of Maxima and Minima algebraically. The reasoning of LAGRANGE has been principally followed, in the most important propositions belonging to this part of the subject.

His rules of computation are the same with those of LEIBNITZ, and all the writers on Fluxions. It was the demonstration only of these rules, and not the rules themselves, which needed to be improved.

30. DEF. A quantity is said to be *constant*, when it is always of the same value throughout any calculation: A *variable* quantity is that which may have any value within certain limits: An *arbitrary* quantity is that of which the value may be supposed, in any calculation, either greater or less than any given finite quantity: In the language of Algebra, a function containing only one variable quantity, is called a *Maximum*, when the variable quantity can neither be increased nor diminished, without the value of the function being thereby made less than it was before: And it is called a *Minimum*, when the variable quantity can neither be increased nor diminished, without the value of the function being thereby made greater than it was before.

31. DEF. Variable quantities being denoted by the last letters of our alphabet, the same letters, with an accent placed above them, are used to express arbitrary quantities, by which those variable magnitudes are supposed to be increased or diminished.

Thus $z - z'$, $z$, $z + z'$ express different successive values of the variable quantity represented by $z$.

32. Def. If, in any algebraic function, containing one or more variable quantities, for each of the variable quantities be substituted the sum of that quantity and its arbitrary increment, and the function be then expanded, that part of the expansion which contains the first powers only of the several increments, is called the *Derivative*, or *First Derivative* of the function; that containing the second powers of the increments, multiplied by 1 . 2, is called the *second Derivative;* and that containing the third powers, multiplied by 1 . 2 . 3, is called the *third Derivative*, and so on.

Thus $z'$ is derivative of $z$; $2xx'$ is the derivative of $x^2$.

## Prop. I.

33. *Problem.* To find the derivative of any power of a variable quantity.

Let $z$ represent any variable quantity; it is required to find the derivative of $z^{\pm\frac{m}{n}}$.

The expansion of $(z+z')^{\pm\frac{m}{n}}$ is (Art. 28.)

$$z^{\pm\frac{m}{n}} \pm \frac{m}{n} z^{\pm\frac{m}{n}-1} z' + \&c.$$

And, because the second term is the only one

which contains the first power only of $z'$, the derivative of $z^{\pm\frac{m}{n}}$ is (Art. 32.)

$$\pm \frac{m}{n} z^{\pm\frac{m}{n}-1} z'.$$

Hence the derivative of any power of a variable quantity is found by the following rule:

Multiply the proposed power by a number which is equal to its index, and then diminish its index by unity, and multiply the result by the derivative of the variable quantity.

It is manifest that the second derivative is obtained from the first, the third from the second, and so on, by the same rule.

34. Cor. 1. The derivative of the sum of any number of powers of variable quantities is the sum of the derivatives of the several powers.

Let

$$x^n + z^m + \&c. = V;$$

then the derivative of $V$ is

$$n\,x^{n-1}\,x' + m\,z^{m-1}\,z' + \&c.$$

35. Cor. 2. The derivative of

$$(a + bz \pm cz^2 + \&c.)^m$$

is

$$m.(a + bz + cz^2 + \&c.)^{m-1}.(b + 2cz + 3\,dz^2 + \&c.)\,z'.$$

Let

$$a + bz + cz^2 + dz^3 + \&c. = x;$$

∴ (Art. 33.)

$$b z' + 2 c z z' + 3 d z^3 z' + \&c. = x';$$

and the derivative of

$$(a + bz + cz^3 + dz^2 + \&c.)^m$$

is equal to the derivative of $x^m$; i. e. (Art. 32.) to $m x^{m-1} x'$, or to

$$m.(a+bz+cz^2+\&c.)^{m-1}.(b+2cz+3dz^2+\&c.).z'.$$

## Prop. II.

36. *Problem.* To find the derivative of the product of two or more powers of variable quantities.

1. Let it be required to find the derivative of $x^m . z^n$.

The product of $(x + x')^m . (z + z')^n$, or of

$$\left\{x^m + m x^{m-1} x' + m.\frac{m-1}{2} x^{m-2} x'^2 + \&c.\right\} \times$$

$$\left\{z^n + n z^{n-1} z' + n.\frac{n-1}{2} z^{n-2} z'^2 + \&c.\right\}$$

is

$$x^m z^n + x^m . n z^{n-1} z' + x^m . n.\frac{n-1}{2} z'^2 + \&c.$$

$$+ z^n . m x^{m-1} x' + m x^{m-1} . n z^{n-1} . x' . z'$$

$$+ z^n . m . \frac{m-1}{2} x^{m-2} x'^2;$$

$\therefore$ (Art. 32.) the derivative of $x^m z^n$ is

$$x^m . n z^{n-1} . z' + z^n . m x^{m-1} . x'.$$

2. In the same manner the derivative of $x^m z^n y^p$ may be shewn to be

$$x^m z^n . py^{p-1} . y' + x^m y^p . n z^{n-1} z' + z^n y^p . m x^{m-1} x'.$$

And the derivative of two, or more, powers of variable quantities is found by this rule;

Multiply the derivative of each of the powers by the product of the rest, and take the sum of the resulting products.

Or, take the derivative of the given product, upon the several suppositions that each of its factors is variable, whilst the rest are constant; and add the results.

37. Cor. The derivative of

$$(a + bx + cx^2 + \&c.)^m \times (A + Bz + Cz^2 + \&c.)^n$$

is

$$(a + bx + cx^2 + \&c.)^m . (A + Bz + Cz^2 + \&c.)^{n-1} . (B + 2Cz + \&c.)\, z' + (A + Bz + Cz^2 + \&c.)^n . m(a + bx + cx^2 + \&c.)^{m-1} . (b + 2cx + \&c.)\, x'$$

(Art. 34.) and is, therefore, found by the same rule as that of $x^m z^n$.

## Prop. III.

38. *Problem.* To find the derivative of a fraction, the numerator and denominator of which are two variable quantities.

Let it be required to find the derivative of $\frac{x^m}{z^n}$.

The quotient of $\frac{(x + x')^m}{(z + z')^n}$, or

$$\frac{x^m + m x^{m-1} x' + m \cdot \frac{m-1}{2} x^{m-2} x'^2 + \&c.}{z^n + n x^{n-1} z' + n \cdot \frac{n-1}{2} z^{n-2} z'^2 + \&c.}$$

is the same as the product of

$$\left(x^m + m x^{m-1} x' + m \cdot \frac{m-1}{2} \cdot x^{m-2} x'^2 + \&c.\right) \times$$
$$\left(z^n + n z^{n-1} z' + n \cdot \frac{n-1}{2} \cdot z^{n-2} z'^2 + \&c.\right)^{-1}$$

or of

$$\left(x^m + m x^{m-1} x' + m \cdot \frac{m-1}{2} x^{m-2} x'^2 + \&c.\right) \times$$
$$(z^{-n} - z^{-2n} \cdot n z^{n-1} z' + \&c.),$$

(Art. 19.); which product is

$$x^m z^{-n} - x^m \cdot z^{-2n} \cdot n z^{n-1} z' + \&c.$$
$$+ z^{-n} \cdot m x^{m-1} x';$$

∴ (Art. 32.) the derivative of

$$\frac{x^m}{z^n} \text{ is } z^{-n} \cdot m x^{m-1} x' - x^m z^{-2n} \cdot n z^{-1} \cdot z',$$

$$\text{or } \frac{z^n \cdot m x^{m-1} x' - x^m n z^{n-1} \cdot z'}{z^{2n}}.$$

Hence the derivative of a fraction, the numerator and denominator of which are powers of two different variable quantities, is found by this rule;

Multiply the derivative of the numerator by the denominator, and also the derivative of the denominator by the numerator; subtract the latter product from the former, and divide the result by the square of the denominator.

39. Cor. The derivative of a fraction of the form $\frac{(a+bx+cx^2+\&c.)^m}{(A+bz+Cz^2+\&c.)^n}$ is found by the same rule.

## Prop. IV.

40. *Theorem.* If the function of a variable quantity be a maximum, or if it be a minimum, its first derivative is equal to nothing.

Let $z^m$ be any * function of a single variable quantity.

The expansion of

$(z+z')^m$ is $z^m + m z^{m-1} z' + m \cdot \frac{m-1}{2} z^{m-2} z'^2 +$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^{m-3} z'^3 + \&c.$$

and the expansion of

$(z-z')^m$ is $z^m - m z^{m-1} z' + m \cdot \frac{m-1}{3} z^{m-2} z'^2 -$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^{m-3} z'^3 + \&c.;$$

therefore (Art. 28.)

$$(1.)\ z^m - (m z^{m-1} z') + \left(m \cdot \frac{m-1}{2} z^{m-2} z'^2\right) - \&c.$$

* The word *function* is used in the sense of the definition in Art. 1.; so that $z$ may represent binomial or polynomial quantities; such, for example, as $ax-x^2$, $ax+x+\frac{b^2}{x}$, &c.; and $z'$ is the derivative of those quantities; $z^m$ may likewise represent $(ax-x^2)^{\frac{1}{2}}$; $\frac{(x+a)^2}{x}$, &c.

(2.) $z^m$, and

(3.) $z^m + (m z^{m-1} z') + \left( m \cdot \frac{m-1}{2} z^{m-2} z'^2 \right) + \&c.$

represent any three consecutive values of the function: let $V$ be the value of $z^m$ when it is a maximum, or when it is a minimum; then, (Art. 30.), if $V$ be a maximum, it must always be greater both than the first, and than the third of the three consecutive values; and if it be a minimum, it must always be less, both than the first and than the third of those values, however little $z'$ be, and whether it be positive or negative.

Let $z'$ be taken (Art. 29.) such that $m \cdot z^{m-1} z'$ shall be greater than the sum of the succeeding terms, in both the expansions; therefore the sum of all the terms but the first will then be positive or negative, accordingly as the second term $m z^{m-1} z'$ is positive or negative; that is, accordingly as $z'$ is positive or negative, for the sign of $m z^{m-1}$ is necessarily the same in both the expansions; whether, therefore, $z'$ be positive or negative,

$$+ (m z^{m-1} z') \text{ and } - (m z^{m-1} z')$$

have, necessarily, contrary signs: the three consecutive values, therefore, upon the supposition here made, will be of the form,

$$(1.) \quad V \pm m z^{m-1} z' \pm \delta.$$
$$(2.) \quad V.$$
$$(3.) \quad V \mp m z^{m-1} z' \pm \delta.$$

$\delta$ being a quantity less than $m z^{m-1} z'$; so that the first or third value of the function is greater or less

than the second, accordingly as $mz^{m-1}z'$ has a positive or negative sign. It is manifest, therefore, that as long as $mz^{m-1}z'$ has any magnitude, $V$ can neither be greater both than the first and third value, nor less both than the first and third value; that is, it can neither be a maximum, nor a minimum, unless $mz^{m-1}z'$ be equal to nothing: if, therefore, the function be either a maximum, or a minimum, its first derivative, $mz^{m-1}z'$, is equal to nothing.

41. COR. 1. When the value of the function is a maximum, its second derivative is negative; and when it is a minimum, its second derivative is positive.

For, whether the function $V$ be a maximum or a minimum, its first derivative (Art. 40.) is equal to nothing; therefore $V$ is greater than

$$V+0+m.\frac{m-1}{2}z^{m-2}z'^2+$$

$$m.\frac{m-1}{2}.\frac{m-1}{3}z^{m-3}z'^3+\&c.$$

when it is a maximum, and less than the sum of that same series, when it is a minimum, however little $z'$ be; let $z'$ be taken (Art. 29.) such, that

$$m.\frac{m-1}{2}.z^{m-2}z'^2$$

shall be greater than the sum of the succeeding terms of the expansion; then, since $z'^2$ is necessarily positive, $V$ is greater or less than

$$V + 0 + m \cdot \frac{m-1}{2} z^{m-2} z'^2 + \&c.$$

accordingly as

$$m \cdot \frac{m-1}{2} \cdot z^{m-2}$$

is negative or positive; if it be necessarily greater, that is, if it be a maximum,

$$m \cdot \frac{m-1}{2} z^{m-2} z'^2$$

must be negative; and if it is a minimum,

$$m \cdot \frac{m-1}{2} z^{m-2} z'^2$$

must be positive; therefore (Art. 32.) the second derivative must, in the former case, be negative, and it must be positive in the latter case.

42. Cor. 2. But, if the second and third terms of the expansion be each equal to nothing, the value $V$, of the function, when it is a maximum, must be greater than

$$V + 0 + 0 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^{m-3} z' + \&c.;$$

and it must be less than the sum of that series, when it is a minimum: As before, let $z'$ be taken such, that

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^{m-3} z'^3$$

shall be greater than the sum of all the succeeding terms of the expansion; then, since the sign of

$z'^3$ depends upon that of $z'$, and is the same with it, $V$ cannot always be greater than the value of that series, nor always less than it, unless

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^{m-3} z'^3$$

be equal to nothing; that is, (Art. 32.) unless its third derivative be equal to nothing.

43. Cor. 3. In the same manner it may be shewn, if the $n$ first derivatives of the function be each equal to nothing, $n$ being an even number, that the $(n+1)^{\text{th}}$ derivative of the function must also be equal to nothing, if the function either be a maximum or a minimum; and that if it be a maximum, the $(n+2)^{\text{th}}$ derivative must be negative; but that if it be a minimum, the $(n+2)^{\text{th}}$ derivative must be positive.

## Prop. V.

44. *Problem.* To determine the conditions under which the aggregate of several functions, each being the function of one variable quantity only, is a maximum or a minimum.

Let $X \pm Y \pm Z \pm$ &c. be the aggregate of the functions, which is to be a maximum or a minimum, $X$ being a function of $x$, $Y$ of $y$, $Z$ of $z$, &c.

1. It is manifest that $X+Y+Z$, &c. all the terms being positive, will be greatest, when $X$, $Y$, $Z$, &c. are each a maximum, taken separately;

and least, when each of these quantities, taken separately, is a minimum.

But, in determining the maximum value of the aggregate, care must be taken lest the minimum value of any one of the functions, taken separately, be joined with the rest; and, likewise, when the minimum value of the aggregate is sought, none of the maximum values of the several functions, taken separately, must be joined with the rest.

2. The binomial $X-Y$ is greatest, when $X$ is a maximum and $Y$ a minimum; and it is least, when $X$ is a minimum and $Y$ a maximum.

3. The trinomial $X+Y-Z$ is greatest, when $(X+Y)$ is a maximum and $Z$ a minimum; and it is least, when $(X+Y)$ is a minimum and $Z$ a maximum.

Also, $X-Y-Z$ is greatest, when $X$ is greatest and $Y+Z$ least; and it is least, when $X$ is least and $Y+Z$ greatest.

The solution of this case is, therefore, reduced to that of the two former cases; and the same method is applicable, to discover when the aggregate of any number of such functions is a maximum or a minimum.

The following example is given in order to illustrate the above proposition. The remaining examples will be taken so as that the determination of the maximum or minimum shall depend

upon the solution either of a simple or of a quadratic equation.

EXAMPLE.

Let it be required to determine when the aggregate of the two functions

$$(x^3 - 3x^2 - 3x) + (z^4 - 8z^3 + 18z^2 - 8z)$$

is a maximum.

Call the former function $X$, and the latter $Z$; the first derivative of

$$X = 3x^2 x' - 6xx' - 3x',$$

which (Art. 40.) must be equal to nothing; therefore

$$x^2 - 2x - 1 = 0.$$

The two roots of this equation are $1 + \sqrt{2}$, and $1 - \sqrt{2}$; if $1 + \sqrt{2}$ be substituted in $(6x - 6) \cdot x'^2$, the second derivative of $X$, the result is positive; therefore, (Art. 41.) when $x$ has this value, the function $X$ is a minimum; if the other root, $1 - \sqrt{2}$ be substituted in the second derivative of $X$, the result is negative; this, therefore, must (Art. 41.) be the value of $x$, in order that the function $X$ shall be a maximum.

Again, the first derivative of $Z$ is

$$4z^3 z' - 24z^2 z' + 36zz' - 8z';$$

therefore, whether $Z$ be a maximum, or a minimum,

$$z^3 - 6x^2 + 9z - 2 = 0.$$

The three roots of this equation are 2, $2+\sqrt{3}$, and $2 - \sqrt{3}$; and when 2 is substituted for $z$ in the second derivative of $Z$, i. e. in

$$(12z^2 - 48z + 36) \cdot z'^2,$$

the result is negative; and it is positive if either of the other roots be substituted for $z$ in the second derivative of $Z$; therefore (Art. 41.) 2 is the value of $z$, which renders the function $Z$ a maximum. Therefore the two values of $x$ and $z$, which must be taken, in order that $X + Z$ may be a maximum, are $1 - \sqrt{2}$ and 2; not $1 + \sqrt{2}$, and $2 \pm \sqrt{3}$ which would make that aggregate a minimum; nor yet $1 - \sqrt{2}$ and $2 \pm \sqrt{3}$; nor $1 \pm \sqrt{2}$ and 2. If the proper values of $x$ and $z$ be substituted, the value of $X + Z$, when it is a maximum, will be found to be $3 + 4\sqrt{2}$.

## SCHOLIUM.

It appears, from Art. 40, that, whether the value of a function be greatest, or least, its first derivative is equal to nothing; if, therefore, there be several values of the variable quantity of the function, which answer this condition, of rendering its derivative equal to nothing, some of them may be such as make the function a maximum, others such as make it a minimum: and it may

be determined whether any particular value $(a)$ of the variable quantity, thus found, belong to the greatest, or the least, value of the function, either by means of Art. 41, as in the preceding example, or by successively substituting in the function $a-\delta$, $a$ and $a+\delta$, for the variable quantity: for then, if the first and last result be both less than the second, it is manifest that the value $(a)$ renders the function a maximum; if the first and last of these results be both greater than the second, $(a)$ renders the function a minimum; and if, of the first and last results, the one be real and the other imaginary, the function may be at once a maximum and a minimum; a maximum in one respect, and a minimum in another.

But if there be no value, of the variable quantity, which will make the first derivative of the function equal to nothing; or, in other words, the first derivative being equated with nothing, if the roots of that equation be all impossible, it is plain that the function does not admit either of a maximum or a minimum.

It may also be determined whether any value $(a)$ of the variable quantity, found according to Art. 40, render its function a maximum, or a minimum, by substituting for it $a-\delta$ in the first derivative: for, this being done, if the first derivative thus become positive, the whole variation of the function may be considered as positive, previously to its having the value, which is either greatest or least; that value must, therefore, in this

case, be a maximum, since the function was gradually *increasing*, before it arrived at that magnitude: on the contrary, $a-\delta$ being substituted for the variable quantity in the first derivative, if the result be negative, it is evident that $(a)$ renders the function a minimum; but if, when $a-\delta$ and $a+\delta$ have been severally put for the variable quantity in the first derivative, the results have the same sign; that is, the derivative being equated with nothing, if that equation have an even number of equal roots, each equal to $(a)$, then $(a)$ will not, properly speaking, give the function a value which is a maximum, or a minimum; for, in that case, if the function be increasing before it had that value, it will manifestly go on increasing after having acquired it; and, if it be decreasing, it will go on decreasing. The same discovery might be made by the application of Art. 43. Thus it appears, that though the proposition stated in Art. 40, be true, its converse is not universally true.

Further, the language of Algebra is not so perfect as always to express, clearly and completely, all the limitations of any proposed question; and the equation, derived from the propositions implied in a problem, may have a root which shall indicate a construction, or an operation, inconsistent with the conditions given, and the grounds on which the reasoning was built: the solution, thus obtained, will not then be true; although the root of the equation be real and

positive. It is necessary, therefore, to ascertain whether the answer to a question respecting maxima or minima, and, generally, whether the result of any other application of Algebra to Geometry, be so circumstanced. This remark was made and exemplified by THOMAS SIMPSON, and it may be seen more fully explained by CARNOT, in his *Geometry of Position.*

## PROP. VI.

45. *Theorem.* When a function of two or more independent quantities is a maximum, or when it is a minimum, if its first derivatives be obtained, upon the several suppositions that each of the variable quantities, in its turn, is the only variable quantity in the function, each of those derivatives is equal to nothing.

First, let $V$ be a function of the two independent variable quantities $x$ and $y$; and let $P \cdot x'$ be the derivative of $V$, when $y$ is constant, and $Q \cdot y'$ the derivative of $V$, when $x$ is constant: if $Q \cdot y'$ be put equal to nothing, such a value of $y$ will be obtained, in terms of $x$ and known quantities, as will render $V$ a maximum or a minimum (Art. 40.) in respect of $y$, whatever be the value of $x$; since the two variable quantities $x$ and $y$ are independent of each other; this value of $y$ is, therefore, a function of $x$; let its derivative be $q \cdot x'$; substitute the value of $y$, thus obtained, in the given function, which will, therefore, become a function of $x$; call

that function $X$; wherefore, the given function is already a maximum or a minimum, as far as regards $y$; and will be absolutely a maximum, if $X$ be a maximum; and absolutely a minimum, if $X$ be a minimum; that is, if the first derivative of $X$ be equal to nothing, and its second derivative be negative, $V$ will be a maximum; and if the first derivative of $X$ be equal to nothing, and its second derivative be positive, $V$ will be a minimum, (Art. 40, 41.) Now it is manifest, $y$ being considered as a function of $x$, that the first derivative of $X$ will be $P \cdot x' + Q \cdot q x'$; which is, therefore, in the case either of a maximum or a minimum, equal to nothing; and the part $P x'$ has been shewn to be equal to nothing; wherefore $Q \cdot q x'$, or $Q \cdot y'$ is also equal to nothing.

(2.) Let $W$ be a function of three independent variable quantities $x$, $y$, and $z$; suppose $z$ to be the only variable quantity, and thus find the value of $z$, in terms of $x$ and $y$, and of known quantities; which will render the function a maximum or a minimum, as far as regards $z$; when this value is substituted in the given function, it will become a function of $x$ and $y$ only; and, thus, this case will be reduced to the former.

In the same manner, the proposition may be shewn to be true, when it respects a function of four, or more, independent variable quantities.

The proposition may also be readily proved *ex absurdo:* for if any of the derivatives, taken upon

the supposition that only one quantity is variable, be not equal to nothing, it is evident that the function may be greater, if it is to be a maximum, or less, if it is to be a minimum, whatever may be the values of the other quantities.

Or, by reasoning as in Art. 40, it may be proved, that whenever any given function of $x$, $y$, $z$, &c. is a maximum, or a minimum, the whole of its first derivative

$$P.x' + Q.y' + R.z' + \&c$$

must be equal to nothing; but

$$P.x' + Q.y' + R.z' + \&c.$$

cannot necessarily be equal to nothing, whatever $x'$, $y'$, and $z'$ be, and whatever signs belong to them, unless $P.x'$, $Q.y'$, $R.z'$, &c. be separately equal to nothing: and (Art. 36.) the derivative

$$P.x' + Q.y' + R.z', \&c.$$

is the aggregate of the derivatives of the given function, taken upon the several suppositions described in the proposition.

46. Cor. It may be shewn, as in Art. 41, that when ($V$) any given function of the independent variable quantities $x$, $y$, $z$, &c. is a maximum, the aggregate of the terms in which $x'$ $y'$, $z'$, &c. have two dimensions, in the expansion resulting from the substitution of

$$x + x', y + y', z + z', \&c.$$

in $V$, is negative; and that it is positive, when $V$ is a minimum.

## SCHOLIUM.

In the preceding proposition, the variable quantities contained in the function, of which the greatest, or the least, value is sought, are supposed to be independent of each other: but if there exist any relations between them, expressed by one or more equations, as many of the variable quantities, as is possible, must be eliminated by means of those equations; and then the greatest, or the least, value must be investigated, of the function containing the rest of the variable quantities.

Again, if the number of variable quantities, assumed in the calculation, exceed the number of the conditions of the question, it is evident that any functions, of each of these quantities, may be considered as constant; provided that these arbitrary conditions, so introduced, together with the real conditions of the question, do not exceed in number the variable quantities.

It may here, also, be remarked, that when the maximum, or minimum, value of a function is sought, it is not always necessary to take the derivative of the function in order to find that value. If an equation be obtained upon the supposition that the function is of a *given* magnitude, it will sometimes appear, from the form of the

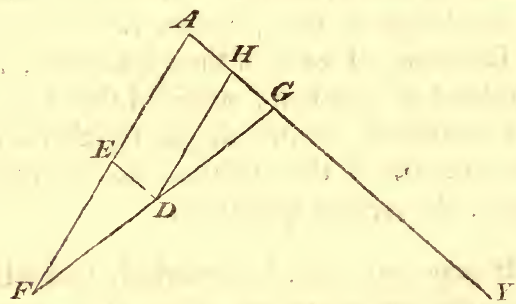
algebraical result, what is the greatest, or least, magnitude, which the function can have.

The following Examples of the application of the method of investigating Maxima and Minima algebraically, are taken from the propositions of the First Part of this Treatise.

## Example I.

To find the least triangle which can be contained by two straight lines, inclined to each other at a given angle, and a third straight line, which passes through a given point between them.

Let $AF$ and $AY$ be two straight lines, inclined




to each other at a given angle, and $D$ a given point between them, through which the straight line $FDG$ passes: the triangle $AFG$ is required to be the minimum.

Let $DE$ be drawn parallel to $AY$; and let the given line $AE$ be denoted by $a$, the given line $ED$ by $b$, and $AF$ by $x$:

Then, (E. 4. 6.)

$$FE : ED :: FA : AG;$$

$$\text{i. e. } x - a : b :: x : AG.$$

Wherefore $AG = \dfrac{bx}{x-a}$. But it follows, from E. 23. 6, that triangles which have equal vertical angles have to one another the same ratio as the rectangles contained by their sides. Therefore, the triangle $AFG$ is the least, when the rectangle $AF \times AG$ is the least; so that the quantity $\dfrac{bx}{x-a} . x$ is to be a minimum; therefore, (Art. 40.),

$$\frac{2x . x' . (x-a) - x^2 . x'}{(x-a)^2} = 0;$$

$$\therefore 2x^2 - 2ax - x^2 = 0;$$

$$\text{i. e. } x^2 - 2ax = 0;$$

$$\therefore x = 2a,$$

as in Art. 3. Part I; and the triangle $= 4ab$.

In order to prove that this value of $x$ renders the triangle the required minimum, let $2a - \delta$, $2a$, and $2a + \delta$ be successively substituted for $x$ in $\dfrac{x^2}{x-a}$; and the results are

$$4a + \frac{\delta^2}{a - \delta}, \; 4a, \text{ and } 4a + \frac{\delta^2}{a + \delta};$$

wherefore, the triangle is less when $x$ has the

value $2a$, than when it has any other value either greater or less; that is, it is then the least that it can be.

The solution may also be obtained upon the principle mentioned at the conclusion of the last Scholium: for let it be required to make the triangle equal to a given rectangle, represented by $2bc$; the area of the triangle, as before, is $\frac{bx^2}{x-a}$;

$$\therefore \frac{bx^2}{x-a} = 2bc;$$

$$\therefore x^2 = 2cx - 2ac;$$

$$\therefore x^2 - 2cx + c^2 = c^2 - 2ac = (c-2a) \cdot c;$$

$$\therefore x - c = \pm \sqrt{c-2a} \cdot \sqrt{c},$$

$$\text{and } x = c \pm \sqrt{c-2a} \cdot \sqrt{c}.$$

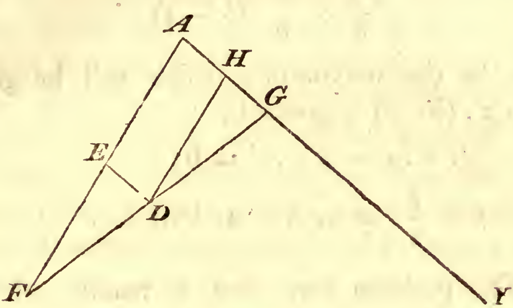
Whence, it is manifest that $c$ cannot be less than $2a$; otherwise the value of $x$ becomes impossible; and the triangle has the least value, when $c$ has the least value; therefore, when the triangle is the required minimum, $x = 2a$, and the triangle itself is equal to $4ab$, as before.

### Example II.

To find the greatest parallelogram which can be inscribed in a given triangle, so as to have the vertical angle of the triangle for one of its angles.

Let $AFG$ be the given triangle; the parallelo-

gram *AEDH*, which is inscribed in it, and has




the vertical angle *A* for one of its angles, is required to be the maximum.

It follows from E. 23. 6, that equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each; therefore the parallelogram *AEDH* will be the greatest when the rectangle $EA \times AH$, or $AE \times ED$, is the greatest.

Let *AF* be denoted by $a$, *FG* by $b$, *GA* by $c$, and *FD* by $x$:

Then, (E. 4. 6.)

$$FG : GA :: FD : DE;$$
$$\text{i.e. } b : c :: x : DE;$$

$\therefore\ DE = \dfrac{c}{b} \cdot x$ ; and in the same manner it may be shewn, that

$$DH = \frac{AF}{FG} \times DG = \frac{a}{b} \cdot (b - x).$$

Wherefore

$$\frac{c\,x}{b} \times \frac{a}{b} \cdot (b - x)$$

is to be the maximum; which will be greatest when $x \cdot (b - x)$ is greatest;

$$\therefore\ b \cdot x' - 2\,x \cdot x' = 0;$$

$$\therefore\ x = \frac{b}{2}, \text{ as in Art. 4. Part I.}$$

The problem may, also, be readily solved by supposing $b\,x - x^2$ to be equal to a given rectangle, as was done in the preceding example.

## Example III.

To find the greatest of all equiangular and isoperimetrical parallelograms.

Let $2\,b$ denote the common perimeter, and $x$ one of the sides of any of the parallelograms; therefore $b - x$ will denote the other side, and, as in the preceding example, the parallelogram will be greatest when $x\,(b - x)$ is greatest; it will, therefore, as in Example 2, be greatest, when $x = \frac{b}{2}$; i. e. when the figure is equilateral, as in Art 6. Part I.

Cor. If the angle $A$, of the parallelogram, be varied as well as the proportion of the sides, the parallelogram will be greatest (Art. 45.) when its sides are all equal, and when $A$ is at the same time

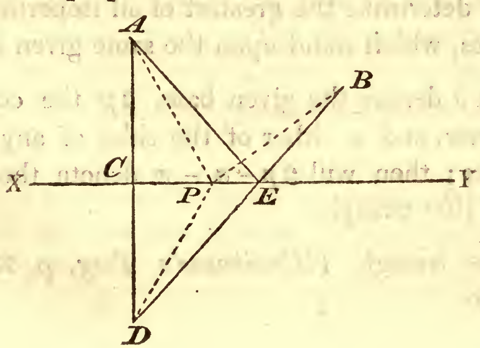
greatest; i. e. when the figure is a square: For, if $S$ and $S'$ be the two sides about the angle $A$, the surface of the figure is expressed by $S . S' \sin A$; and the sine of $A$ is greatest when $A$ is a right angle. But if the sides remain constant, and the angle, at which they are inclined to each other, be varied, it is manifest, from the expressed value of the parallelogram, that it is greatest, when it is a rectangle; which agrees with Art. 6. Part I.

### Example 4.

To find a point, in an indefinite straight line, from which, if two straight lines be drawn to two given points, without the indefinite line and on the same side of it, their aggregate shall be a minimum.

Let $XY$ be the indefinite straight line, and $A$, and $B$, two given points, without it; a point $P$ is to be found, in $XY$, such that $AP + PB$ shall be a minimum.

Let the perpendiculars drawn from $A$ and $B$ to



$XY$ be denoted by $a$ and $b$, the portion of $XY$, included between them, by $c$, and $CP$ by $x$; then

$$AP = \sqrt{a^2 + x^2}, \text{ and } BP = \sqrt{b^2 + (c - x)^2};$$

wherefore $\sqrt{a^2+x^2} + \sqrt{b^2+(c-x)^2}$ is to be the minimum;

$$\therefore \frac{x \,.\, x'}{\sqrt{a^2 + x^2}} - \frac{(c - x) \,.\, x'}{\sqrt{b^2 + (c - x)^2}} = 0;$$

$$\therefore \frac{x}{\sqrt{a^2 + x^2}} = \frac{c - x}{\sqrt{b^2+(c-x)^2}};$$

$$\therefore \frac{x^2}{a^2 + x^2} = \frac{(c - x)^2}{b^2 + (c - x)^2};$$

$$\therefore b^2 x^2 + x^2 \,.\, (c-x)^2 = a^2 \,.\, (c-x)^2 + x^2 \,.\, (c-x)^2;$$

$$\therefore b^2 x^2 = a^2 \,.\, (c - x)^2;$$

$$\therefore b x = a \,.\, (c - x);$$

$$\therefore (a + b) \,.\, x = a c;$$

$$\therefore a + b : a :: c : x;$$

$\therefore b : a :: c - x : x$, as in Art. 12. Part I.

## Example 5.

To determine the greatest of all isoperimetrical triangles, which stand upon the same given base.

Let $a$ denote the given base, $2p$ the common perimeter, and $x$ either of the sides of any of the triangles; then will $2p - a - x$ denote the other side of that triangle.

The triangle (*Woodhouse's Trig.* p. 23.) is equal to

$$\sqrt{p \cdot (p-a) \cdot (p-x) \cdot (a+x-p)},$$

which is, therefore, to be the maximum; and it will plainly be greatest when $(p-x) \cdot (a+x-p)$ is greatest;

$$\therefore (p-x) \cdot x' - (a+x-p) \cdot x' = 0;$$

$$\therefore 2p - a - 2x = 0;$$

$$\therefore x = \frac{2p-a}{2};$$

wherefore, the triangle is greatest when it is isosceles, as in Art. 16. Part I.; where this conclusion is deduced from another proposition, with which it is connected. It may, however, be seen demonstrated directly and geometrically, in two different manners, by Pappus and Legendre.

### Example 6.

To find the greatest of all isoperimetrical triangles.

It is evident, from the preceding example, that of whatever magnitude the base of a triangle of given perimeter may be, the triangle cannot be a maximum, unless it be isosceles; let it, therefore, be required to find the greatest isosceles triangle of a given perimeter.

Let $2p$ denote the given perimeter, and $2x$ the base of an isosceles triangle of that perimeter; then will $\frac{2p-2x}{2}$, or $p-x$ denote the side of the

triangle, and $\sqrt{(p-x)^2-x^2}$ the perpendicular drawn from its vertex to the base; therefore the triangle is equal to $\sqrt{(p-x)^2-x^2}\,.\,x$, or to $\sqrt{p^2-2px}\,.\,x$, which is to be the maximum.

$$\therefore \sqrt{p^2-2px}\,.\,x' - \frac{px'}{\sqrt{p^2-2px}}\,x\,.=0\,;$$

$$\therefore p^2-2px=px\,;$$

$$\therefore 3px=p^2\,;$$

$$\therefore x=\frac{p}{3}, \text{ and } 2x=\frac{2p}{3}\,;$$

therefore, when the triangle is greatest, it is equilateral, as in Art. 25. Part I.

Or, the given perimeter being denoted, as before, by $2p$, let $x$ and $y$ denote the two sides of any triangle of that perimeter, and $z$ and $s$ the two segments of the base made by the perpendicular drawn to it from the vertex; $z$ denoting that which is adjacent to the side denoted by $x$. Then

$x+y+z+v=2p$, and $\frac{1}{2}(z+v)\,.\,\sqrt{x^2-z^2}$

is to be the maximum; and since there are four unknown quantities, and only two conditions belonging to the question, two more conditions may be introduced. Let, therefore, $\sqrt{x^2-z^2}$ and its equal, the quantity $\sqrt{y^2-v^2}$, be supposed to be each of them invariable, whatever be the values of $x$ and $z$, and of $y$ and $v$; so that the derivatives of these quantities shall be equal to nothing.

And, because $\frac{1}{2}(z+v)\,.\,\sqrt{x^2-z^2}$ is to be a

maximum, and the derivative of $\sqrt{x^2 - z^2}$ is equal to nothing,

$$(z' + v') . \sqrt{x^2 - z^2} = 0,$$

(Art. 36. and 40.)

$$\therefore z' + v' = 0.$$

Also, because $x + y + z + v$ is equal to the invariable quantity $2p$,

$$\therefore x' + y' + z' + v' = 0;$$

but $z' + v'$ has been shewn to be equal to nothing;

$$\therefore x' + y' = 0.$$

Whence $\frac{x'}{z'} = \frac{y'}{v'}$. Again, from the conditions introduced,

$$\frac{y . y' - v . v'}{\sqrt{y^2 - v^2}} = \frac{x . x' - z . z'}{\sqrt{x^2 - z^2}} = 0;$$

$$\therefore y . y' - v . v' = 0; \text{ and } x . x' - z . z' = 0;$$

$$\therefore \frac{z}{x} = \frac{x'}{z'} = \frac{y'}{v'} = \frac{v}{y};$$

$$\therefore z : v :: x : y.$$

Wherefore (E. 3. 6.) the perpendicular bisects the vertical angle of the triangle, when the triangle is greatest; also the two triangles, into which it is divided by the perpendicular, have (E. 26. 1.) their other sides and angles respectively equal. Hence the original equation becomes $2x + 2z = 2p$; and $z . \sqrt{x^2 - z^2}$ is now to be the maximum; or, since $z = p - x$, $(p - x) . \sqrt{2px - p^2}$ is to be the maximum;

$$\therefore\ (p-x)\,.\,\frac{p\,.\,x'}{\sqrt{2px-p^2}}-\sqrt{2px-p^2}\,.\,x'=0\,;$$

$$\therefore\ p^2-px=2px-p^2\,;$$

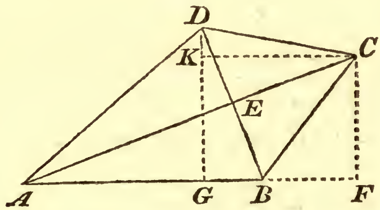
$$\therefore\ 3x=2p,$$

and $x=\dfrac{2p}{3}$ as before.

## Example 7.

Of all quadrilateral rectilineal figures, which have equal perimeters, to find the greatest.

Let $ABCD$ be a quadrilateral rectilineal figure




of the given perimeter, the area of which is to be a maximum; join $B$, $D$, and $A$, $C$; then it is manifest, from Example 5, that the two sides $AB$, $AD$, and also the two sides $CB$, $CD$, must be equal to each other; and, therefore, (E. 8. and 4. 1.) $AC$ must bisect $BD$, at right angles, in the point $E$.

Let the perimeter be denoted by $2a$, $AE$ by $x$, $EB$ by $y$, and $EC$ by $z$; therefore (E. 47. 1.)

$$AB+BC=(x^2+y^2)^{\frac{1}{2}}+(z^2+y^2)^{\frac{1}{2}}=a,$$

and the figure

$$ABCD = (x + z) \cdot y,$$

which is to be a maximum.

Thus, there are three unknown quantities, and only two conditions implied in the problem; therefore, either

$$(x^2 + y^2)^{\frac{1}{2}}, \text{ or } (z^2 + y^2)^{\frac{1}{2}},$$

may be supposed a constant quantity, and its derivative will, consequently, be equal to nothing.

Whence (I.) $z \cdot z' + y \cdot y' = 0; \quad \therefore z' = -\frac{y \cdot y'}{z};$

(II.) $x \cdot x' + y \cdot y' = 0; \quad \therefore x' = -\frac{y \cdot y'}{x};$

(III.) $x \cdot y' + y \cdot x' + z \cdot y' + y \cdot z' = 0;$

therefore,

$$x \cdot y' - \frac{y^2 \cdot y'}{x} + z \cdot y' - \frac{y^2 \cdot y'}{z} = 0;$$

$$\therefore x^2 z - y^2 z + z^2 x - y^2 x = 0;$$

$$\therefore x^2 z + z^2 x = y^2 z + y^2 x;$$

$$\therefore (z + x) \cdot zx = (z + x) \cdot y^2;$$

$$\therefore zx = y^2,$$

$$\text{and } x : y :: y : z;$$

wherefore (E. 6. 6.) the two triangles *AEB*, *CEB* are similar to each other, and the angle *ABC* is a right angle, as is also the angle *ADC*.

Again, the two triangles *ABC*, *ADC* being (E. 8. 1.) equal to each other, the figure *ABCD* is the double of *ABC*, and is, therefore, equal to

the rectangle $BA \times BC$; the two sides of which $AB$, $BC$ are, together, equal to the given semi-perimeter $a$. The question is, therefore, reduced to the finding of the greatest rectangle contained by a given perimeter; but (Example 3.) such a rectangle is greatest when its sides are equal. Wherefore the greatest quadrilateral rectilineal figure, contained by a given perimeter, is a square.

## SCHOLIUM.

The same conclusion may be arrived at, if no two sides of the quadrilateral figure $ABCD$ be supposed equal to each other, by drawing $DG$ and $CF$ perpendicular to $AB$, and $CK$ parallel to it, and making $AG$, $GD$, $GB$, $BF$, $CF$ the unknown quantities; the perimeter and area of the figure may then be expressed in terms of these quantities; the expression for the perimeter will be found to contain three radical quantities, which, as the number of unknown variable quantities exceeds the number of conditions, implied in the problem, by three, may each be supposed a constant quantity. Eliminating, upon this supposition, the several derivatives of the unknown quantities, it will first appear that $AG=0$, and that, therefore, the angle $DAB$ is a right angle; it will next be found that $BF=0$, and that, therefore, the angle $ABC$ is also a right angle; and lastly, as in the example, the area of the figure will be expressed

by $AB \times BC$, $AB + BC$ being the half of the given perimeter; whence, it is manifest, that when the area is greatest the figure is a square. And by the same method it may be shewn that, when the number of sides and the perimeter of any polygon are both given, the area is greatest when the figure is equilateral and equiangular.

### Example 8.

Of all triangles standing upon the same given base, and on the same side of it, and having their vertical angles equal and given, to find that which has the greatest perimeter.

The converse of E. 21. 3. having been proved *ex absurdo*, it will be manifest that the locus of the summits of all the triangles, standing upon the given base, and having their vertical angles each equal to the same given angle, will be the arch of the segment of a given circle, described about any one of the triangles, of which the given base is the chord.

Let the remainder of the circumference be bisected; let the straight line joining the point of bisection and the vertex of any one of the triangles, be denoted by $z$, and the chord of half the bisected arch by $a$; also, let $b$ denote the given base of the triangle, and $x$ and $y$ its two sides: Then $x + y$ is to be a maximum. But (*Simson's* Euclid, Prop. D. Book VI.)

$$a\,(x + y) = b\,z\,;$$

Whence, it is manifest, that $(x+y)$ is greatest when $z$ is greatest: but $z$ is greatest when it passes through the center; that is, when the triangle is isosceles its perimeter is a maximum, as in Art. 69. Part I.

The trigonometrical solution of the problem is as follows:

Let $ABC$ be any one of the triangles, on the given base $BC$, and having its vertical angle $A$ of the given magnitude: Let

$$D = 180^\circ - A, \text{ and let } x = \sin B\,;$$
$$a = \sin D;\ b = \cos D.$$

$$\text{Then, } AC = 2x$$
$$AB = 2\,.\,\sin C$$
$$= 2\,.\,\sin\,(D - B)$$
$$= 2\,.\,(a\sqrt{1-x^2} - bx).$$

And $AB+AC$, i. e. $2x + 2\,(a\sqrt{1-x^2}-bx)$ is to be a maximum;

$$\therefore\ (1-b)\,x' - \frac{a\,x\,x'}{\sqrt{1-x^2}} = 0\,;$$

$$\therefore\ \frac{1-b}{a} = \frac{x}{\sqrt{1-x^2}};$$

$$\text{i. e. } \tan \tfrac{1}{2}\,D = \tan B\,;$$

$$\therefore\ \angle B = \tfrac{1}{2}\,\angle D.$$

Whence (E. 32. 1.) the triangle is isosceles, when its perimeter is the greatest.

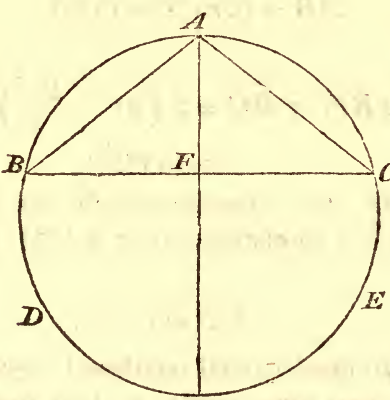
## EXAMPLE. 9.

Of all triangles inscribed in the same given circle, to find that which has the greatest perimeter.

By the preceding example, the inscribed triangle of the greatest perimeter is necessarily isosceles.

Let, therefore, $ABC$ be any isosceles triangle inscribed in the circle $ADE$; and let $AF$ be drawn perpendicular to its base $BC$; therefore (E. 26. 1.) $AF$ bisects $BC$, and is, therefore, (E. 1. 3. Cor.) when produced, a diameter of the circle.

Let $2r$ denote the diameter of the given circle,




$x$ the straight line $AF$, $y$ the straight line $BF$ or $CF$; then (E. 8. 6.)

$$y^2 = 2rx - x^2; \therefore y = (2rx - x^2)^{\frac{1}{2}},$$

and $AB^2 = 2rx$; $\therefore AB = (2rx)^{\frac{1}{2}}$.

And

$2AB + 2 . y$, or $AB + y$, i. e. $(2rx)^{\frac{1}{2}} + (2rx - x^2)^{\frac{1}{2}}$ is to be a maximum;

$$\therefore \frac{r . x'}{(2rx)^{\frac{1}{2}}} + \frac{(r-x) . x'}{(2rx - x^2)^{\frac{1}{2}}} = 0;$$

$$\therefore \frac{r}{(2rx)^{\frac{1}{2}}} = \frac{x - r}{(2rx - x^2)^{\frac{1}{2}}};$$

$$\therefore \frac{r^2}{2rx} = \frac{x^2 - 2rx + r^2}{2rx - x^2};$$

$$\therefore 2r^3x - r^2x^2 = 2rx^3 - 4r^2x^2 + 2r^3x$$

$$3r^2x^2 = 2rx^3$$

$$\therefore 3r = 2x.$$

Whence,

$$AB = (2rx)^{\frac{1}{2}} = (3r)^{\frac{1}{2}},$$

and

$$2BF, \text{ or } BC = 2\left(3r^2 - \frac{9r^2}{4}\right)^{\frac{1}{2}}$$

$$= (3r^2)^{\frac{1}{2}};$$

wherefore, the triangle which has the greatest perimeter is equilateral, as in Art. 71. Part I.

## Example 10.

Of all quadrilateral rectilineal figures inscribed in the same given circle, to find that which has the greatest perimeter.

It is manifest, from Example 8, that at least

one of the diagonals of the greatest inscribed quadrilateral figure must divide it into two isosceles triangles; and, therefore, the other diagonal is (E. 8. and 4. 1. and E 1. 3. Cor.) necessarily a diameter of the circle.

Let $a$ denote the diameter of the given circle, and $x$ and $y$ the two sides of the inscribed figure which are not supposed equal to each other; then (E. 31. 3. and 47. 1.) $x^2 + y^2 = a^2$, whence $y = (a^2 - x^2)^{\frac{1}{2}}$; and $2x + 2y$, or $x + y$; that is, $x + (a^2 - x^2)^{\frac{1}{2}}$ is to be a maximum.

$$\therefore x' - \frac{x \cdot x'}{(a^2 - x^2)^{\frac{1}{2}}} = 0;$$

$$\therefore (a^2 - x^2)^{\frac{1}{2}} = x;$$

$$\therefore a^2 - x^2 = x^2;$$

$$\therefore a^2 = 2x^2 = x^2 + y^2;$$

$$\therefore x^2 = y^2, \text{ and } x = y;$$

wherefore, the inscribed figure which has the greatest perimeter is a square, as in Art. 72. Part I.

## Example 11.

Of all triangles standing upon the same given base, and having their vertical angles each equal to the same given angle, to find the greatest.

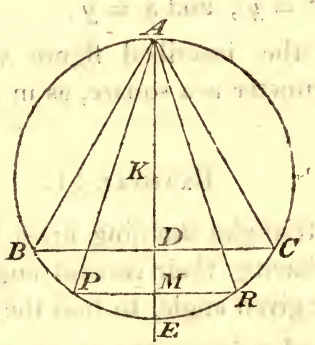
Let $x$ and $y$ denote the two sides of any one of the triangles, and $a$ the tabular sine of the given vertical angle; then the area of the triangle is equal to $\frac{axy}{2}$, which will manifestly be greatest

when $xy$ is greatest; but (Example 9.) the aggregate of $x$ and $y$ is greatest, when the triangle is isosceles, i. e. when $x=y$; and (Example 3.) a square is greater than any other rectangle of equal perimeter; wherefore the rectangle $xy$ cannot be greater, than it is when $x=y$; and the greatest triangle standing on the given base, and having its vertical angle also given, is, therefore, that which is isosceles, as in Art. 73. Part I.

## Example 12.

To find the greatest of all triangles which can be inscribed in the same given circle.

The greatest inscribed triangle is necessarily isosceles (Example 11.); let, therefore, $APR$ be




any isosceles triangle, inscribed in the given circle $ABC$; and draw $AM$ perpendicular to $PR$.

Then, if the diameter be denoted by $2r$, $AM$ by $x$, and $PM$ by $y$, as in Example 9, $y =$

$(2rx - x^2)^{\frac{1}{2}}$; and the area of the triangle is the half of the rectangle contained by $AM$ and $PR$, and is, therefore, equal to $yx$, or to $(2rx - x^2)^{\frac{1}{2}}.x$, which is to be the maximum;

$$\therefore (2rx - x^2)^{\frac{1}{2}}.x' + \frac{(r-x).x'.x}{(2rx - x^2)^{\frac{1}{2}}} = 0;$$

$$\therefore 2rx - x^2 = x^2 - rx;$$

$$\therefore 3r = 2x;$$

whence, as in Example 9. the maximum sought is an equilateral triangle, according to Art. 76. Part I.

## Example 13.

To find the greatest quadrilateral rectilineal figure which can be inscribed in a given circle.

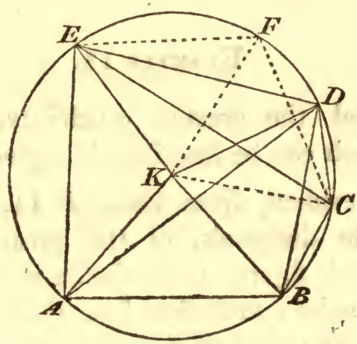
It is evident, from Example 11, that at least one of the diagonals, of the greatest inscribed quadrilateral figure, must divide it into two isosceles triangles; and, therefore, the other diagonal must be a diameter of the given circle, and must cut the former diagonal at right angles; wherefore an inscribed quadrilateral figure, of this kind, is equal to the rectangle contained by the diameter of the circle and the other diagonal; and it will be greatest when that diagonal is greatest; that is, when the diagonal is equal to the diameter of the circle; in which case (E. 6. 4.) the figure is a square, as in Art. 77. Part I.

### EXAMPLE 14.

To find the greatest of all pentagons which can be inscribed in a given circle.

Let *ABCDE* be a pentagon inscribed in the given circle *AEDCB*, of which the center is *K*; it may be shewn, by means of Example 11, that at least four sides of the figure must be equal to each other, when it is a maximum.

For, first, if *A*, *D* and *B*, *D* be joined, the two sides *AE* and *ED* must (Example 11.) be equal




to each other, and also the two sides *BC*, *CD*; join *K*, *A* and *K*, *B*, and *K*, *C* and *K*, *D*, and *K*, *E*; then (E. 8. 1.) the two triangles *EKA* and *EKD* are equal, and also the two triangles *BKC*, *DKC*; again, if the arch *EDC* be bisected in *F*, and *E*, *F* and *F*, *C*, and *C*, *E* and *K*, *F* be joined, the triangle *EFC* is greater (Example 11.) than the triangle *EDC*; and if the triangle *EKC* be

added to both, it is evident that the two equal triangles *EKF, FKC* are, together, greater than the two triangles *EKD, DKC*; wherefore four times the triangle *EKF*, or *FKC*, is greater than the aggregate of the four triangles *AKE, EKD, DKC, CKB*; but the arch *EF*, or *FC*, is a fourth part of the whole arch *AEFDCB*; wherefore, the side *AB* remaining the same, the inscribed pentagon is greatest when its four remaining sides are equal to each other.

Let the pentagon, therefore, be supposed to be made up of four equal triangles and the triangle *AKB*; and let $a$ denote the vertical angle of each of the equal triangles, $x$ its sine; $b$ the vertical angle *AKB*, $y$ its sine, and $r$ the radius *KA*; then $4a$ and $b$ will have equal sines, but if the one sine be positive, the other will be negative; also the area of the pentagon is equal to $4r^2x + r^2y$, which is to be a maximum; or $4x \sim \sin 4a$ is to be a maximum.

$$\text{Sin } 4a = (4x - 8x^3) \cdot (1 - x^2)^{\frac{1}{2}}.$$

(*Woodhouse's Trig.* p. 41.)

$$\therefore x \sim (x - 2x^3) \cdot (1 - x^2)^{\frac{1}{2}}$$

is to be a maximum; therefore

$$x' \sim \left((1-6x^2) \cdot (1-x^2)^{\frac{1}{2}} \cdot x' - (x-2x^3) \cdot \frac{x \cdot x'}{(1-x^2)^{\frac{1}{2}}}\right) = 0;$$

$$\therefore (1-x^2)^{\frac{1}{2}} = (1-6x^2) \cdot (1-x^2) - x^2 + 2x^4$$
$$= 1 - 8 \cdot x^2 \cdot (1 - x^2);$$

$$\therefore 8 \,.\, x^2 \,.\, (1 - x^2) = 1 - (1 - x^2)^{\frac{1}{2}};$$

$$\text{i. e. } 8 \,.\, \sin^2 a \,.\, \cos^2 a = 1 - \cos a.$$

But

$$\sin a \,.\, \cos a = \tfrac{1}{2} \sin 2a,$$

(*Woodhouse's Trig*. p. 41.)

$$\text{and } 1 - \cos a = 2 \,.\, \sin^2 \left(\frac{a}{2}\right),$$

(*Woodhouse's Trig*. p. 13.)

$$\therefore 2 \sin^2 2a = 2 \sin^2 \frac{a}{2};$$

$$\therefore \sin 2a = \sin \frac{a}{2};$$

wherefore the two arches $2a$ and $\frac{a}{2}$ must be supplements to each other, and $2a + \frac{a}{2} = 180$;

$$\text{i. e. } \frac{5a}{2} = 180;$$

$$\therefore a = \frac{360}{5};$$

$$\text{but } 4a + b = 360;$$

$\therefore \; b = \dfrac{360}{5}$, and the pentagon is equilateral.

In the same manner it might be shewn, that the perimeter of the equilateral pentagon inscribed in a circle, is greater than that of any other pentagon inscribed in the same circle.

## SCHOLIUM.

It is manifest that the process, by which the preceding problems have been solved, is the same as that which would have been employed, if they had been treated fluxionally. In reality, from the principles laid down in Sect. I. and II. Part II, all the rules of the Method of the Fluxions have been deduced by Lagrange: and by the help of one of the elementary propositions of that method, the solutions of some of the problems, which have here been proposed as examples, may be shortened.

In all treatises on the Method of Fluxions, or the Differential Method, it is shewn, that if $A$ be a circular arch, of which $r$ is the radius, $s$ the sine, and $c$ the cosine,

$$\dot{A} = r \cdot \frac{\dot{s}}{c}.$$

Hence, in Example 8. if a circle be supposed to be described about the triangle, which, having a given base and a given vertical angle, is to have its perimeter a maximum, and if $A$ be the given angle, and $B$ and $C$ the two angles at the base, the sides will be the doubles of the sines of the opposite angles;

$$\therefore \ \sin B + \sin C, \text{ i.e. } \sin B + \sin (A+B),$$

is to be a maximum;

$$\therefore \dot{B}\,\frac{\cos B}{r} - \dot{B}\times\frac{\cos(A+B)}{r} = 0^{*};$$

$$\therefore \cos B - \cos C = 0;$$

$$\therefore B = C;$$

that is, the triangle is isosceles.

In the same manner, in Example 9, where the perimeter of the triangle inscribed in a given circle is to be a maximum, if $A$, $B$ and $C$ be the three angles of the triangle, it may be shewn, since the triangle must necessarily be isosceles, that

$2 . \sin B + \sin A$, i. e. $2 \sin B + \sin(180° - 2B)$

is to be a maximum;

$$\therefore 2\dot{B} \cos B - 2\dot{B} . \cos(180° - 2B) = 0;$$

$$\therefore \cos B = \cos(180° - 2B);$$

$$\therefore B = 180° - 2B;$$

$$\therefore B = \frac{180°}{3} = 60°.$$

From which it is plain, that the triangle must be equilateral.

Lastly, in Example 14, since it is there shewn that

$$4 \sin a \sim \sin 4a$$

is to be a maximum; therefore,

$$4\dot{a} \cos. a \sim 4\dot{a} \cos 4a = 0;$$

$$\therefore \cos a = \cos 4a;$$

---

* The cosines of $B$ and $A+B$ have necessarily contrary signs.

$$\therefore\ a + 4a = 360^\circ;$$

$$\therefore\ a = \frac{360}{5}:$$

Whence it is concluded, as before, that the greatest of all pentagons, which can be inscribed in a given circle is equilateral.

## EXAMPLE 15.

To draw the shortest tangent to a given circular arch, which shall be terminated by the semi-diameters, produced, that pass through the extremities of the arch. Let $A$ be the given arch: then, it is manifest that any tangent, terminated by the two semi-diameters produced, is divided by the point of its contact into two segments, which are the trigonometrical tangents of the parts $P$, and $Q$, into which that same point divides the given arch. If, therefore, $a$ be the trigonometrical tangent of $A$, $x$ of $P$, and if the radius of the circle be denoted by unity, then (*Woodhouse*, p. 30.) $\frac{a-x}{1+ax}$ is the trigonometrical tangent of $Q$; and

$$x + \frac{a-x}{1+ax}, \text{ or } \frac{ax^2+a}{ax+1}$$

is to be a minimum; $\therefore\ \frac{x^2+1}{ax+1}$ is to be a minimum;

$$\therefore\ 2ax^2x' + 2xx' - ax^2x' - ax' = 0;$$

$$\therefore ax^2 + 2x - a = 0;$$

$$\therefore x = \frac{a - x}{1 + ax};$$

that is, the trigonometrical tangent of *P* is equal to the trigonometrical tangent of *Q*, or (as in Art. 103.) the given arch *A* is bisected, when the tangent required to be drawn is to be a minimum.

### Example 16.

If two straight lines touch a given circle, to draw the shortest straight line which can be terminated by them and touch the given circle.

If the two given tangents be parallel to each other, the angles which they make, with the tangent which is to be a minimum, are, together, (E. 29. 1.) equal to two right angles; and they are equal (E. 32. 1.) to the supplement of a given angle, if the two given tangents meet each other; the aggregate of these angles is, therefore, in both cases, equal to a given angle; and consequently, (E. 32. 1.) the angle, subtended by the tangent, which is to be a minimum, at the center of the circle is also a given angle. So that the problem is reduced to that which was solved in Example 15: and the tangent, required to be drawn is a minimum, when as in Art. 56, it is bisected in the point of its contact.

If, therefore, **AI* and *AL* be the two given

* See the figure in p. 79.

tangents, touching the given circle $BFC$, the shortest tangent $IL$ makes with them an isosceles triangle: And, if $PO$ be any other tangent terminated by the given tangents, since the perimeter of the triangle $APO$ is the double of $PO$ and of the given line $AB$, taken together, the perimeter is manifestly least when $PO$ is least; that is, when the triangle is isosceles.

In the same manner it may be shewn, that the quadrilateral figure contained by two given tangents to a circle and two other straight lines touching the circle, in two points lying between the given tangents, has the least perimeter, when those two straight lines are bisected, each in its point of contact.

## SCHOLIUM.

The problem, solved in the above example, is evidently reduced to the dividing of a given circular arch into two such parts, that the aggregate of their tangents shall be a minimum; it has appeared that the given arch must, in that case, be bisected: and if it were shewn that when a circle is divided into $2n$ equal parts, the aggregate of the tangents of the parts is greater, than when the circle is divided into $2n$ parts which are not all equal; it might thence be concluded, that a regular polygon, described about a given circle, has a less perimeter than any other polygon of

the same number of sides, described about the same circle, which is not equilateral and equiangular. It is manifest, at once, from the above conclusion, that when the circle is divided into pairs of equal arches, the aggregate of all the tangents of the parts may be made less, by bisecting the aggregate of any two contiguous arches, which are not equal: and, by following the method of reasoning indicated in Art. 28. Part I. it may be shewn, that the several divisions which must be made, in order to render the aggregate of the tangents less and less, will bring the arches nearer and nearer to a state of equality. A proof of the proposition, which asserts that a regular polygon, inscribed in a circle, has a greater perimeter than another polygon of the same number of sides, inscribed in the same circle, which is not regular, may be deduced, in a similar manner, from the conclusion, that when a given circular arch is bisected, the aggregate of the sines of its parts is a maximum.

ON

# MAXIMA AND MINIMA.

## PART III.

### ON THE STRUCTURE OF THE CELLS OF BEES.

It has been asserted, that to have read the celebrated work of *Cervantes*, in the original, is an ample compensation for the labour of learning the Spanish tongue; and the peculiar excellence, in its kind, of that inimitable performance may seem to justify the assertion. But if the pleasure arising from the perusal of a single work of fiction can be thought to counterbalance the pains of acquiring a foreign language, it may with greater justice and seriousness be maintained, that the time and study consumed in acquiring a competent knowledge of the Mathematics, are more than repaid by the satisfaction to be derived from the speculations upon which we are now to enter. They terminate in conclusions so astonishing, so curious,

and so strongly evincing the operative wisdom of the Creator of the world, that even the Grecian mathematician, when he introduces the subject of the Œconomy of the Bee, seems to forget the grave and simple style of his science in his admiration of the geometry of Providence.

But the light, with which we have been indulged by a gracious Revelation, whilst it enables us more clearly and more certainly to assign an effect like this to its true Author, has not rendered useless the contemplation of such an instance of divine contrivance. On the contrary, we shall do well, from the same source with PAPPUS, to improve our reverence for the Being, who has endued an insect with the power of working such wonders, and to share the gratitude, which he expresses, for having been enabled to comprehend them.

Natural religion, when it is rightly cultivated, will be found most powerfully to administer to our faith and to our devotion. May it not, indeed, be questioned, whether an unlettered person, who is unacquainted with such instances of the divine contrivance as are exhibited in the mechanism of an eye, the structure of a honey-comb, or the laws of gravitation, and the planetary motions, can have a veneration for the Deity, of the same kind and degree, as that of the Christian philosopher who has diligently studied Nature? There is, undoubtedly, enough that is obvious, in the visible world, to inspire the least cultivated, and

the least acute, of understandings, with sufficient awe of our common Maker; nor can it be disputed, that the smallest portion of genuine religious principle is infinitely more valuable than the most accurate and extensive knowledge is, without it. Still, if we mistake not, there is a peculiar sense of the character of God, which flashes upon the mind, from the consideration of such instances of his wisdom as those which have been alleged; it is the present reward of a successful enquiry into his works, and, perhaps, the fore-taste of some of those intellectual enjoyments, which will constitute the happiness of our future state.

The observation has already been made, that that part of the Collections of Pappus, in which he treats of the Cells of Bees, is now imperfect. The principal proposition, which was wanting, was demonstrated, in a synthetic form, by MACLAURIN; it is here reprinted, nearly as he gave it in the Philosophical Transactions; and an anlytical solution of it is subjoined. Several subordinate steps have, also, been supplied, which are not to be found either in Pappus, or in Maclaurin; but which lead, by an easier and less interrupted ascent, to the great object of the investigation.

The two methods of mathematical reasoning, which, in the former parts of this Work, have been carefully kept distinct, are here applied, in some measure, conjointly. But this mixt use of them may be said, if it need any apology, to be called

for by the subject; and it does not at all interfere with the main design of this treatise; which is to compare, in the instances adduced, the respective merits of Algebra and Geometry, in the investigation of Maxima and Minima.

## DEFINITIONS.

1. A *Prism* is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms: And, if the sides of a prism be perpendicular to the plane of its base, it is called a *Right Prism.*

2. The *Axis* of a right prism, of which the base is a regular polygon, is a straight line drawn through the center of that polygon, perpendicular to the plane of its base.

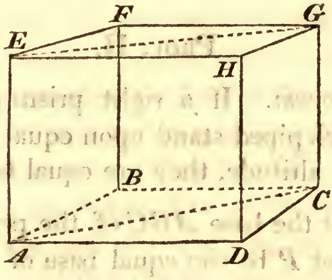
3. A *Parallelepiped* is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel: and a *Rectangular Parallelepiped* is one which is contained by rectangles, having their planes perpendicular to each other.

### Prop. I.

4. *Theorem.* If, from the angular points of a given polygon, equal straight lines be drawn, all

of them perpendicular to its plane, and their extremities be joined, the planes bounded by them will contain a right prism, of which the given polygon is the base.

Let *ABCD* be the given polygon; from the




angular points *A, B, C, D,* let there be drawn (E. 12. 11.) the equal straight lines *AE, BF, CG, DH,* each perpendicular to the plane *AC*; and let the points *E, F,* and *F, G,* and *G, H,* and *H, E* be joined by the straight lines *EF, FG, GH,* and *HE*; the solid figure *AG* is a right prism.

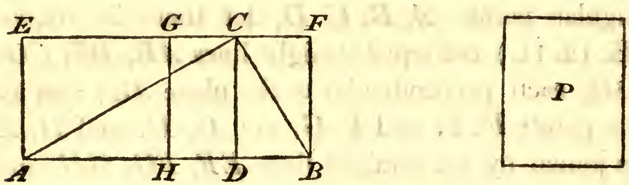
For (E. 6. 11.) *AE* and *BF* being equal and parallel straight lines, *EF* (E. 33. 1.) is equal and parallel to *AB*; therefore *AEFB* is a parallelogram, and (E. 18. 11.) it is perpendicular to the plane *AC*. In the same manner *BFCG, DHGC,* and *DHEA* may each be shewn to be a parallelogram, and to be perpendicular to the plane *AC*. Join *E, G* and *A, C*; then (E. 33. 1.) *EG* is parallel to *AC*, and the plane *EFG* is parallel to the plane *AC* (E. 15. 11.); for the same reason the plane *EHG* is also parallel to *AC*; wherefore

(E. 14. and 4. 11.) *EFG* and *EHG* are in the same plane, and this plane is parallel to *ABCD*; therefore (Art. 1. Part III.) *AG* is a prism; and because the planes *AF*, *BG*, *DG*, and *DE* are each perpendicular to *AC*, it is a right prism.

## Prop. II.

5. *Theorem.* If a right prism and a rectangular parallelepiped stand upon equal bases, and be of the same altitude, they are equal to one another.

First, let the base *ABC* of the prism be a triangle, and let *P* be the equal base of the parallele-

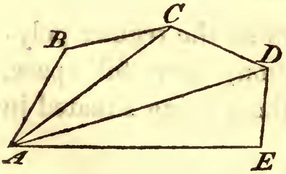


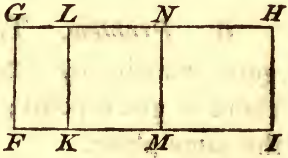

piped of the same altitude as the prism. The two solid figures are equal.

For, through *C* draw (E. 32. and 12. 1.) *ECF* parallel, and *CD* perpendicular, to *AB*. Bisect (E. 10. 1.) *AB* in *H*, and through *A*, *H*, and *B* draw (E. 11. 1.) *AE*, *HG*, and *BF* each perpendicular to *AB*: then are *ABFE*, *AHGE*, and *HBFG* rectangular parallelograms; and (E. 36. 1.) the parallelograms *AG* and *HF* are equal; wherefore *AG* is the half of *AF*, and (E. 41. 1.) the triangle *ACB* is also the half of *AF*; therefore

*AG* is equal to the triangle *ACB*; but, by the hypothesis, the triangle *ACB* is equal to the rectangle *P*; therefore *AG* is equal to *P*; and it is manifest, (Art. 4. Part III,) that the prism standing on *ACB* may be divided into two prisms standing on the bases *ADC* and *BDC* respectively; and since (E. 34. 1.) the triangle *AEC* is equal to the triangle *ADC*, and the triangle *CDB* to the triangle *CFB*, therefore the two prisms standing on *ADC* and *BDC* will be equal to two right prisms, of the same altitude, standing on *AEC* and *BFC*, each to each: wherefore the prism standing on *ACB* will be the half of the prism, or parallelepiped, of which *AEFB* is the base, and, therefore, will be equal to the parallelepiped standing on *AHGE*; but (E. 31. 11.) the parallelepiped standing on *AHGE* is equal to that of the same altitude, of which *P* is the base; therefore the prism standing on *ACB* is equal to the parallelepiped standing on *P*.

But if the base *ABCDE* of the prism be a




polygon, let *FGHI* be the equal base of the parallelepiped. Join *A*, *C* and *A*, *D*; to the

straight line $FG$ apply (E. 44. 1.) the rectangle $FGLK$ equal to the triangle $ABC$; and to $LK$ apply the rectangle $KLNM$ equal to the triangle $ACD$; then, since the two whole figures are equal, and the part $FGNM$, of the one, has been made equal to the part $ABCD$, of the other, the remainder $MNHI$ must be equal to the remainder $ADE$.

Again, (Art. 4. Part III.) the prism standing on $ABCDE$ may be divided into prisms, of which the triangles $ABC$, $ACD$, $ADE$, are the respective bases; and the parallelepiped standing on $FGHI$ may be likewise divided into similar solid figures, of which $FL$, $KN$, and $MH$, are the bases; and it has been proved that the prisms standing on $ABC$, $ACD$, and $ADE$ are respectively equal to the parallelepipeds standing on $FL$, $KN$, and $MH$; therefore the whole prism standing on $ABCDE$ is equal to the parallelepiped standing on $FGHI$.

### Prop. III.

6. *Problem.* To determine the regular polygons, which, by juxta-position, may fill space, about a given point; all of them being situated in the same plane.

Let $x$ represent the number of sides of any of the regular polygons, which, placed in contact, may fill space, about a given point; then, since

(Art. 22. Part I.) the angles of a regular polygon are all equal, if $A$ be put for a right angle, one of the angles of that figure will (E. 32. 1. Cor. 1.) be equal to

$$\frac{2x.A - 4.A}{x}, \text{ or to } 2A.\frac{x-2}{x};$$

which quantity, in order that the polygons may fill space about a given point, must be a divisor of $4A$; that is, $\frac{2x}{x-2}$ must be a whole number:

$$\text{but } \frac{2x}{x-2} = 2 + \frac{4}{x-2};$$

wherefore $\frac{4}{x-2}$ must, also, be a whole number; or $x - 2$ must be a divisor of 4. But the only divisors of 4 are 1, 2, and 4; therefore $x - 2$ is equal to 1, or 2, or 4; and $x$ must, therefore, be equal to 3, 4, or 6; i. e. the regular polygons, which may fill space, about a given point, must either have three, four, or six sides; they must, therefore, be either equilateral triangles, squares, or regular hexagons.

## Prop. IV.

7. *Theorem.* If two right prisms of the same altitude have for bases two equal regular polygons, that, of which the base has the greater number of sides, will have less superficies.

For, the lateral surface of a right prism standing upon a regular polygon, consists (Art. 4. Part III.) of as many equal rectangles, each of the same altitude with the prism, as its base has sides; it is, therefore, (E. 1. 2.) equal to a rectangle of that same altitude, and the base of which is equal to the perimeter of the base of the prism; but (Art. 39. Part I.) of two equal regular polygons, the perimeter of that is the less, which has the greater number of sides; wherefore, the lateral surface of the right prism, which stands upon the regular polygon having the greater number of sides, is the less: and the surface of the two ends is, by the hypothesis, equal in both; therefore, if two right prisms, of the same altitude, stand upon equal regular polygons, that, of which the base has the greater number of sides, will have the less surface.

8. Cor. Hence, of all equal prismatic cells, of the same depth, which may fill space, about a given straight line, that of which the base is a regular hexagon, has the least surface.

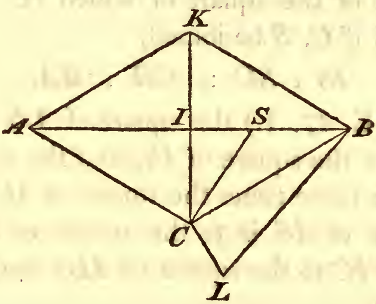
For (Art. 6. Part III.) the only kinds of cells, which may fill space, about a given straight line, are such as have either equilateral triangles, squares, or regular hexagons for their bases; and (Art. 5. Part III.) if their bases and altitudes be equal, each to each, the cell will be equally capacious; but (Art. 7. Part III.) that, which is hexagonal, will have the least surface.

## Prop. V.

9. *Theorem.* The diameters of a rhombus bisect each other at right angles.

Let *ACBK* be a rhombus, and *AB, CK* its diameters; *AB* and *CK* are each bisected at right angles, in their point of intersection *I*.

For (E. Def. 32. 1.) *AK* is equal to *AC*, *AB* is common to the two triangles *CAB, KAB*, and




the base *KB* is equal to the base *CB*; wherefore (E. 8. 1.) the angle *KAB* is equal to the angle *CAB*.

Again, because *AK* is equal to *AC*, and *AI* common to the two triangles *AIK, AIC*, and that the angle *KAI* has been shewn to be equal to the angle *CAI*, therefore (E. 4. 1.) *KI* is equal to *IC*, and the angle *AIK* to the adjacent angle *AIC*; wherefore, *KC* is bisected at right angles

in $I$: And in the same manner $AB$ may be shewn to be bisected at right angles in $I$. Therefore the two diameters $AB$ and $KC$ bisect each other at right angles.

10. COR. 1. If $BKC$ be an equilateral triangle, that is, (E. 15. 4.) if the angle $ACB$ be one of the angles of a regular hexagon, $CB$ is the double of $CI$.

For, $KC$ (Art. 9. Part III.) is the double of $CI$, and $CB$ is, by the hypothesis, equal to $KC$.

11. COR. 2. The same supposition being made, as in the preceding corollary, if $IS$ be made equal to the side of the square of which $IC$ is the diameter, and if $C$, $S$ be joined,

$$IS : SC :: CB : BA.$$

For, (E. 47. 1.) the square of $CS$ is equal to three times the square of $IS$, and the square of $IB$ is equal to three times the square of $IC$; therefore the square of $IS$ is to the square of $SC$ as the square of $IC$ to the square of $IB$; and, therefore, (E. 22. 6.)

$$\begin{aligned} IS : SC &:: IC : IB \\ &:: CB : BA \end{aligned}$$

(E. 15. and 11. 5.); for $CB$ is the double of $IC$ (Art. 10. Part III.), and $BA$ (Art. 9. Part III.) is the double of $IB$.

12. COR. 3. The same supposition and construction being made, as in the preceding corollaries, if $CL$ be drawn (E. 11. 1.) perpendicular to $BC$, and made equal to $IS$, that is, if $LC$ be to

*IC* as the side of a square is to its diameter, then, if *B, L* be joined,

$$BL : CL :: 3 : 1.$$

For, by the hypothesis, the square of *IC* is the double of the square of *IS* or *CL*; wherefore the square of *CB*, which (Art. 10. Part III. and E. 4. 2.) is four times as great as the square of *IC*, is eight times as great as the square of *LC*; therefore (E. 47. 1.) the square of *BL* is equal to nine times the square of *CL*; and (E. 5. Def. 5.) the square of *BL* is to the square of *CL* as 9 to 1; but (E. 20. 6. Cor 1.) this is the duplicate ratio of the respective sides of the two squares; therefore,

$$BL : CL :: 3 : 1.$$

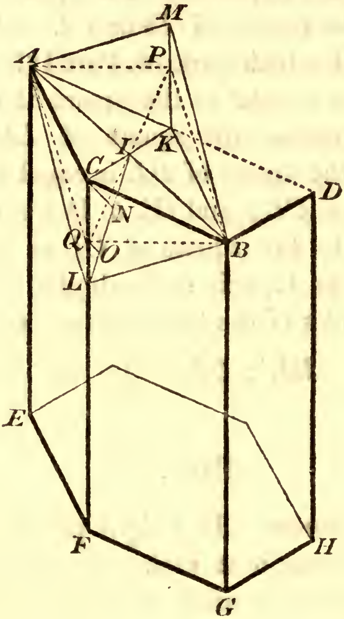
## Prop. VI.

13. *Theorem.* If a right prism, bounded by a regular hexagon at each end, be cut by three planes, forming a solid angle at any point of its axis, and each of them passing through two alternate angles of the hexagon, the capacity of the solid figure, thus formed, shall be equal to that of the prism.

Let *AC, CB* be any two adjacent sides of the hexagon, which bounds the right prism *AH* toward one of its ends; let *K* be the center of that hexagon, and *KM* the axis of the figure; let *P* be any point in the axis, and *PAQB* one of the

planes, which form a solid angle at *P*; and let the plane *PAQB* pass through *A* and *B*.

Join *A*, *B* and *C*, *K*; and let *AB*, *CK* cut




each other in *I*. Then (E. 15. 4.) *ACBK* is a rhombus, and (Art. 9. Part III.) the diameters *AB*, *CK* bisect each other at right angles in *I*. In the plane *PQ*, join *P*, *I*, and *Q*, *I*: Then, because *AK* is equal to *KB*, and (E. Def. 3. 11.) *PK* is perpendicular to *AK* and *KB*, *AP* (E. 4. 1.) is equal to *PB*. Again, because *AI* is equal (Art. 9. Part III.) to *IB*, and *AP* has been proved to be equal to *PB*, and that *PI* is common to the two tri-

angles *AIP* and *BIP*, the angle *AIP* (E. 8. 1.) is equal to the angle *PIB*, and, therefore, each of them is a right angle. In the same manner *AQ* may be shewn to be equal to *QB*, and *QIA*, *QIB* to be right angles; wherefore (E. 14. 1.) *PIQ* is a straight line. And, because in the two triangles *PIK*, *CIQ*, the angle *PIK* is equal (E. 15. 1.) to the angle *CIQ*, and the angles at *K* and *C* are right angles, and that *CI* is equal to *IK*, therefore (E. 26. 1.) *PK* is equal to *CQ*, and *PI* is equal to *IQ*; wherefore, also, (E. 4. 1.) *AQ* is equal to *AP*; and the figure *APBQ* is a rhombus, wherever *P* be taken in the axis *KM*.

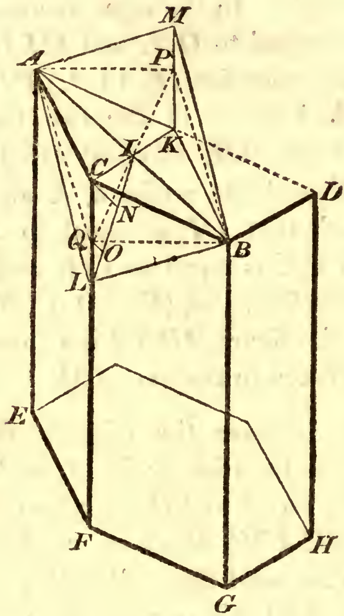
Lastly, because the triangle *BKA* is equal (E. 34. 1.) to the triangle *BCA*, and *PK* has been proved to be equal to *CQ*, therefore (E. 5. 12.) the solid figure *QABC*, which is cut off from the prism, is equal to the solid figure *PABK*; and, therefore, the whole of what is cut off from the prism, by the three rhombs, which form the solid angle at *P*, is equal to the solid space, which they add by their junction; and the capacity of the solid so formed is equal to that of the prism.

## Prop. VII.

14. *Problem.* To determine when the solid, described in the sixth proposition, has the least superficies.

Let *AH* be the original prism, and, the con-

struction remaining as it is described in the pre-




ceding article, in $CF$, the common section of any of the rectangles which bound the prism, take $CL$ to $CI$ as the side of a square is to its diameter; and let $AMBL$ be a rhomb, passing through $A$ and $B$, and cutting $CF$ in $L$; then, $AMBL$ is one of the rhombs which contain the solid angle, when the surface of the figure is a minimum.

For, join $L, I$; let $APBQ$ be any other rhomb passing through $A$ and $B$; and, first, let it meet $CF$, in a point $Q$, which lies between $C$ and $L$.

From $Q$ draw (E. 12. 1.) $QO$ perpendicular to

*IL*. The triangles *LOQ* and *LCI* having a right angle in each, and another angle common, are similar; therefore (E. 4. 6.)

$$LO : LQ :: LC : LI;$$

but, by the construction, and (Art. 11. Part III.)

$$LC : LI :: CB : BA;$$

therefore (E. 11. 5.)

$$LO : LQ :: CB : BA,$$

and (E. 16. 6.) the rectangle contained by *AB* and *LO* is equal to that contained by *LQ* and *CB*.

But (E. 34. and 41. 1. and Art. 9. Part III.) the rhomb *ML* is equal to the rectangle contained by *AB*, *LI*, and the rhomb *PQ* is equal to the rectangle *AB*, *QI*; and (E. 17. 1.) the angle *CQI* is less than a right angle; wherefore (E. 13. 1.) the angle *IQL* is greater than a right angle, and (E. 19. 1.) *IL* is greater than *IQ*; and, therefore, the rhomb *ML* is greater than the rhomb *PQ*, and exceeds it by the difference of the two rectangles *AB*, *LI*, and *AB*, *IQ*; i. e. by a rectangle which has *AB* for its altitude, and the excess of *IL* above *IQ* for its base; and, because (E. 19. 1.) *IQ* is greater than *IO*, *LO* is greater than the excess of *IL* above *IQ*; wherefore, the rhomb *ML* exceeds the rhomb *PQ*, by a rectangle which is less than that contained by *AB* and *LO*.

Again, the excess of the two trapeziums *AEFQ*, *FQBG* above the two trapeziums *AEFL*, *FLBG* is equal to the aggregate of the two equal tri-

angles $BQL$, $AQL$, or to the rectangle (E. 41. 1.) contained by $CB$ and $LQ$, which has been shewn to be equal to the rectangle $AB$, $LO$; therefore the excess of the two trapeziums $AEFQ$, $FQBG$ above the two $AEFL$, $FLBG$ is greater than that of the rhomb $ML$ above the rhomb $PQ$; therefore the rhomb $ML$, together with the two trapeziums $LE$, $LG$, is less than the rhomb $PQ$, together with the two trapeziums $QE$, $QG$.

If, now, a similar construction be made at $D$, and at the angle opposite to $C$, the surface of the solid figure thus formed, may be shewn to be a minimum. And the same mode of proof is applicable, if the point $Q$ lie between $L$ and $F$.

15. Cor. 1. In the solid figure, thus determined, bounded by three equal rhombs and six equal trapeziums, the obtuse angle of the rhomb is equal to the obtuse angle of the trapezium; the acute angle, also, of the one, is equal to the acute angle of the other; and the three plane angles, which form any of the solid angles of the figure, are equal to each other.

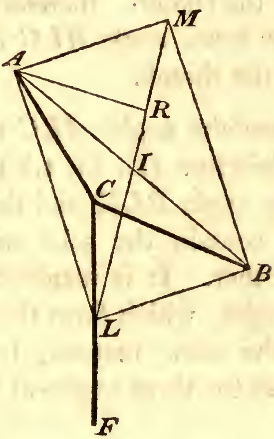
For, let $ALBM$ be one of the equal rhombs; bisect $MI$ in $R$, and join $AR$; wherefore (Art. 9. Part III.) $LI$ is the double of $IR$; also, from the construction,

$$LC : CB :: LI : AB :: IR : IA$$

(E. 15. 5.) because $LI$ is the double of $IR$, and $AB$ is the double of $IA$ (Art. 9. Part III.); wherefore

$$LC : CB :: IR : IA,$$

and the angles $LCB$, $AIR$, are right angles;




therefore (E. 6. 6.) the triangles $BCL$, $ARI$ are similar; and (E. 4. 6.)

$$AR : RI :: BL : CL;$$

but (Art. 12. Part III.)

$$BL : CL :: 3 : 1;$$

therefore $AR$ is equal to three times $RI$; and $RL$ is also equal to three times $RI$; therefore $AR$ is equal to $RL$, and (E. 5. 1.) the angle $RAL$ is equal to the angle $RLA$; and (E. 32. 1.) the angle $ARM$ is equal to the double of $ALM$, that is, to the obtuse angle $ALB$ of the rhomb; but the angle $ARI$ is equal to the angle $BLC$, the two triangles $ARI$, $BLC$ having been shewn to be similar; also, since (E. 8. 1.) the two triangles $ALC$, $BLC$, are equal, the angle $ALC$ is equal to $BLC$, and is, therefore, equal to $ARI$; wherefore

(E. 13. 1.) the angle *BLF* is equal to the angle *ARM*, and consequently to its equal, the obtuse angle *ALB* of the rhomb; therefore, also, (E. 13. and 29. 1.) the acute angle *BLC* is equal to the acute angle of the rhomb.

Again, since the angle *ALC* is equal to the angle *BLC*, therefore (E. 13. 1.) the angle *ALF* is equal to the angle *BLF*, and the three obtuse angles, which contain the solid angle at *L*, are equal to each other. It is manifest, also, that the three acute angles, which form the solid angle at *B*, may, in the same manner, be shewn to be equal, as well as the three angles at the summit *M* of the figure.

16. Cor. 2. The surface of the solid figure thus determined, in the seventh proposition, is less than the surface of a right prism of the same capacity, which has a hexagon for its base.

For the demonstration of the seventh proposition may be applied to the case, in which the rhomb *APBQ*, instead of cutting the hexagonal prism, coincides with the rhomb *AKBC*; so as to shew, that the surface of the solid figure, determined in that proposition, is less than that of the hexagonal prism: and (Art. 13.) the capacities of the two solids are equal to one another*.

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* The surface of the cell, when it is terminated by a hexagon, will be found to exceed its surface when terminated by

## Prop. VIII.

17. *Problem.* To determine, algebraically, when the solid described in the sixth proposition has the least surface; and to compute the angle, at which each of the equal rhombs is, in that case, inclined to the axis of the solid, and also the angles of the rhomb.

It is manifest, from the description of the figure in Art. 13, that the aggregate of the rhomb $ALBM$ and the two trapeziums $AEFL$, $LFGB$, is to be a minimum.

For $CI$, or (Art. 10.) $\frac{1}{2}$ $CB$, put $a$; for $CF$, $b$; and for $CL$, $z$; therefore

$$LI = (LC^2 + CI^2)^{\frac{1}{2}} = (z^2 + a^2)^{\frac{1}{2}};$$

and

$$AB = 2BI = 2(BC^2 - CI^2)^{\frac{1}{2}} = 2(4a^2 - a^2)^{\frac{1}{2}} = 2\sqrt{3}.a;$$

$$\therefore \text{ the rhomb } ML = 2\sqrt{3}\,.\,a \times (z^2 + a^2)^{\frac{1}{2}}.$$

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by a pyramid of the form above described, by $3AB.(IC-IN)$, that is, by

$$3\,AB\,.\frac{(3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{(3)^{\frac{1}{2}}}\,.\,IC, \text{ or } .55 \times AB\,.\,IC \text{ nearly}:$$

and the whole surface of the three terminating rhombs is

$$3AB \times IL = 3\frac{(3)^{\frac{1}{2}}}{(2)^{\frac{1}{2}}}.\,AB\,.\,IC, \text{ or } 3.668. \times AB \times IC \text{ nearly};$$

therefore the excess of the former surface, above the latter, is nearly one-sixth part of the whole surface of the three terminating rhombs.

And the two trapeziums

$$= 2(CG - BCL)$$
$$= 2(2ab - az)$$
$$= 2a(2b - z);$$

$$\therefore 2\sqrt{3}\,.\,a\,.\,(z^2 + a^2)^{\frac{1}{2}} + 2a(2b - z)$$

is to be a minimum,

$$\text{or } \sqrt{3}\,.\,(z^2 + a^2)^{\frac{1}{2}} + 2b - z$$

is to be a minimum;

$$\therefore \sqrt{3}\frac{z\,.\,z'}{(z^2 + a^2)^{\frac{1}{2}}} - z' = 0;$$

$$\therefore \sqrt{3}z = (z^2 + a^2)^{\frac{1}{2}};$$

$$\therefore 3z^2 = z^2 + a^2;$$

$$\therefore 2z^2 = a^2, \text{ and } z^2 = \frac{a^2}{2};$$

$$\therefore z = \frac{a}{\sqrt{2}}, \text{ and } z : a :: 1 : \sqrt{2};$$

$$\text{i. e. } CL : CI :: 1 : \sqrt{2},$$

as the side of a square is to its diameter, according to the construction in Prop. 6.

To compute the angle *ILC*, which is equal to the angle, at which the plane of the rhomb is inclined to the axis of the figure:

In the right-angled triangle *LCI*,

$$CL : CI :: \text{rad.} : \tan \angle\, ILC;$$

$$\text{i. e. } 1 : \sqrt{2} :: \text{rad.} : \tan \angle\, ILC;$$

$$\therefore \log.\,\text{rad.} + \log.\sqrt{2} = 10.1505150$$
$$= \log.\tan 54°.\ 44'.\ 8''.$$

$\therefore$ the angle $ILC$ is an angle of 54°. 44′. 8″.

To compute the angle of the rhomb.

In the right-angled triangle $BCL$,

$$BL : LC :: \text{rad.} : \cos \angle BLC;$$

i. e. (Art. 12. Part III.) $3 : 1 :: \text{rad.} : \cos \angle BLC$;

$$\begin{aligned}\therefore \log. \text{rad.} - \log. 3 &= 10 - .4771212 \\ &= 9.5228788 \\ &= \log. \cos. 70°. 31'. 44''.\end{aligned}$$

Therefore, the acute angle $LBM$ of the rhomb, which (Art. 15. Part III.) has been shewn to be equal to the angle $BLC$, is an angle of 70°. 31′. 44″.

Therefore, the obtuse angle of the rhomb is an angle of 109°. 28′. 16″, which is the double of the angle at which each of the equal rhombs is inclined to the axis of the solid.

## SCHOLIUM.

The fabrication of the system of cells which compose a honey-comb is carried on in strict conformity with the theory which has been established in the preceding articles.

In order that the honey-comb may be compact and strong, and that it may not occupy more space than is absolutely necessary, it is evident that there ought to be no interstices between the cells of which it is composed; it appears, from Art. 6. Part III, that there is a choice of figures, which have this

important property, of filling space about a given point, when placed in contact with each other; and it is shewn, in Art. 7. Part III, that, of all these, the hexagonal prism is of the most œconomical structure. Now this is the model according to which the bee really works.

But a *complete* hexagonal prism is not, in every respect, suited to its purpose. The safety of the grub, and the preservation of the honey in which it is bedded, and which is its nutriment, require that the cell should terminate in a solid angle rather than in a flat surface. Here again there is a choice of figure, and indeed an infinite variety of modes, in which such a termination might be formed; but it is proved, in Art. 14. Part III, that there is one particular mode, that is the most advantageous, as requiring the least quantity of labour and materials: and the cells of the honey-comb were found by MARALDI, upon exact measurement, to terminate in that very angle, which is mathematically ascertained to render the surface the least possible, in such a structure. Further, the surface of the cell, which from its figure thus secures the safety of the embryo, is demonstrated, in Art. 16. Part III, to be less even than that of a complete hexagonal prism of equal capacity.

Another remarkable consequence of the actual construction of the cell, is the simplicity of the component parts of its termination. It may be seen, in Art. 15. Part III. that there are only two

different angles, and those each the supplement of the other, employed in the fabrication. This advantage is not wholly unconnected with the hexagonal form of the cells. If they had not been hexagonal, but had still, for a reason which has been given, ended in pyramids, these pyramids must have been bounded by trapeziums, and not by rhombs; and there would neither have been the same regularity, nor the same facility of construction. A saving of a different kind, also, from that which has been pointed out, accrues from the termination of the cells in these equal solid angles, and from the rows of cells being placed back to back. For, by the combination of two bases of cells belonging to one row, with a third belonging to the same row, the bottom of a new cell is formed, belonging to the opposite row; and the work is strengthened by this junction of opposite cells, and their locking, as it were, into each other. The regularity with which this most ingenious structure is carried on, by so many thousands of insects, labouring together without any plan to imitate, the great delicacy of the workmanship, and the extraordinary exactitude with which three sets of rhombs are continued in three planes, are truly surprizing.

But if such depth of Geometry be manifested in the *form*, there is displayed as wonderful a regard to the principles of chemical science in the *size*, also, of the cell of the honey-comb. For, had its dimensions been larger than they are, the honey, with

which it is filled, would have fermented, and would thus have been spoiled.

Many other circumstances, also, belonging to this portion of Natural History, which do not lie within the sphere of mathematical enquiry, are most worthy of observation.

Such, for example, are the anatomy of the bee, the conformation of the different organs with which it extracts the honey and the wax, and the mechanism of its sting; its habits and propensities, its prognostications of the weather; its prospective industry in laying up stores for winter, and hermetically sealing those cells which are last to be opened; its precautions against cold, which appears to be its greatest physical evil; its strong affections and hatreds; the distribution of labour which takes place in the hive; the modelling of a few cells of larger dimensions for the propagation of drones, and of a still less number, apart from the rest, as nurseries for the queens of future swarms; the singular instinct by which the queen-bee constantly deposits the proper egg in the peculiar kind of cell which is accommodated to it. All these particulars might be enlarged upon, if this were the proper place, and many more, equally curious, might be described, which it is at once delightful and instructive to contemplate.

5/293 Rare